SOLUTIONS TO SAMPLE PROBLEMS FOR QUIZ 1

Problem 1. Write $f(t) = a_n t^n + \cdots + a_1 t + a_0$ where $a_i \in F \subset F(s)$. Consider the polynomial $g(x) = a_n x^n + \cdots + a_1 x + (a_0 - s) \in F(s)[x]$. Then $g(t) = f(t) - s = 0$. We found a polynomial of degree $n = \deg f$ over $F(s)$ which has t as a root. Since $F(t) = F(s)(t)$, $[F(t) : F(s)] = [F(s)(t) : F(s)] \leq n = \deg f$. In particular, the extension is finite, hence algebraic. One can also show that $[F(t) : F(s)] = \deg f$.

(b) From (a) we know that $[F(t) : F(s)] < \infty$. If $F(s)/F$ were algebraic, it would be finite since it is a simple extension. Thus, by the product formula we would have $[F(t) : F] = [F(t) : F(s)][F(s) : F] < \infty$. This would imply that $F(t)/F$ is algebraic, a contradiction.

Problem 2. Let $f(x) = x^4 + 8x^3 + 19x^2 + 12x + 6$. If $a \neq 0, b \in \mathbb{Q}$, we know that $f(x)$ is irreducible if and only if $f(ax + b)$ is irreducible. In an attempt to get rid of the coefficient of x^3, we substitute $x - 2$ in f, we get that $f(x - 2) = x^4 - 5x^2 + 10$ which is irreducible by Eisenstein's criterion with the prime 5. Hence $f(x)$ is also irreducible.

Problem 3. $\alpha = \sqrt[3]{11}$, $K = \mathbb{Q}(\alpha)$.

(a) $[K : \mathbb{Q}] = 3$. The degree $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ is equal to the degree of the minimal polynomial of α over $\mathbb{Q}$. Consider the polynomial $p(t) = t^3 - 11 \in \mathbb{Q}[t]$. It is irreducible over $\mathbb{Q}$ by Eisenstein's criterion with the prime 11, and has α as a root. It follows that $p(t)$ is the minimal polynomial of α over $\mathbb{Q}$ and $[\mathbb{Q}(\alpha) : \mathbb{Q}] = \deg p = 3$.

(b) The extension $K/\mathbb{Q}$ is not normal. It is enough to find an irreducible polynomial in $\mathbb{Q}[t]$ which has a root in K but does not split in K. We take $p(t) = t^3 - 11$ which is irreducible by (a). Let $\omega = e^{2\pi i/3}$. Then $\omega^3 = 1$ so the roots of p in $\mathbb{C}$ are $\alpha, \omega\alpha, \omega^2\alpha$. Now $\alpha \in K$ but $\omega\alpha \notin K$ because $\alpha \in \mathbb{R}$ hence $K = \mathbb{Q}(\alpha) \subset \mathbb{R}$ but $\omega\alpha \notin \mathbb{R}$.

(c) $\mathrm{Gal}(K/\mathbb{Q}) = \{id\}$. Since $K = \mathbb{Q}(\alpha)$, any $\mathbb{Q}$-automorphism of K is determined by its action on α. We know that if α is a root of $p(t) \in \mathbb{Q}[t]$ and σ is any $\mathbb{Q}$-homomorphism then $\sigma(\alpha)$ is also a root of $p(t)$. Take $p(t) = t^3 - 11$. Then, by (b), α is the only root of $p(t)$ in K, hence any automorphism of K must fix α hence fix K (pointwise).

Problem 4. Assume $[L : K] = p$ is prime. Let $\alpha \in L \setminus K$. Then $K \subsetneqq K(\alpha) \subseteq L$ and $[L : K] = [L : K(\alpha)][K(\alpha) : K]$. Since p is prime, one of the factors is p and the other 1. The right factor cannot be 1 since $K(\alpha) \neq K$, hence $[L : K(\alpha)] = 1$ and $L = K(\alpha)$.

Problem 5. (a) *False.* Consider the field $\mathbb{C}$ of complex numbers and let $L/\mathbb{C}$ be a finite algebraic extension. Let $\alpha \in L$ and let $p(t) \in \mathbb{C}[t]$ be its minimal polynomial. Since $\mathbb{C}$ is algebraically closed, every polynomial in $\mathbb{C}[t]$ splits over $\mathbb{C}$, so the only irreducible polynomials are of degree 1. Since

p is irreducible, we have $\deg p = 1$ and its root lies in $\mathbb{C}$. It follows that $\alpha \in \mathbb{C}$ hence $L = \mathbb{C}$ and $\mathbb{C}$ does not have any nontrivial (finite) algebraic extensions.

(b) *False.* For any field K, consider the field $K(t)$ of rational functions over K. Then the extension $K(t) \supset K$ is simple (generated by the element t) but not algebraic, because the element t is not algebraic over K; for any nonzero polynomial p, one has $p(t) \neq 0$.