HOMEWORK #6 IN ALGEBRAIC STRUCTURES 2

We assume in this homework that $\operatorname{char} K \neq 2, 3$.

Problem 6.1. Let $L : K$ be a *quadratic extension* (that is, $[L : K] = 2$). Show that it is a normal extension.

Problem 6.2. Let $L : K$ be a *cubic extension* (that is, $[L : K] = 3$). Let $\alpha \in L \setminus K$ and let $p(t) := \operatorname{Irr}(\alpha, K, t)$ be the minimal polynomial of α.

Show that:

(a) $L = K(\alpha)$,
(b) There exists a monic quadratic polynomial $q(t) \in L[t]$ such that $p(t) = (t - \alpha)q(t)$,
(c) The extension $L : K$ is normal if and only if the polynomial $q(t)$ has a root $\beta \in L$,
(d) If the extension $L : K$ is normal then $\operatorname{Gal}(L/K)$ is the group of cyclic permutations of the set $\{\alpha, \beta, \gamma\}$ where γ is the second root of $q(t)$,
(e) If the extension $L : K$ is not normal then $M := L[t]/(q(t))$ is the normal closure and $\operatorname{Gal}(M/K)$ is the group S_3 of all permutations of $\{\alpha, \beta, \gamma\}$ where β, γ are roots of $q(t)$ in M.

Problem 6.3.

(a) Show that there exists a polynomial $\Delta(c_0, c_1) \in \mathbb{Z}[c_0, c_1]$ with integer coefficients such that for any field K and any polynomial $p(t) = t^2 + c_1 t + c_0 \in K[t]$, we have

$$\Delta(c_0, c_1) = (\alpha_1 - \alpha_2)^2$$

where $\alpha_1, \alpha_2 \in \bar{K}$ are the roots of $p(t)$ (that is, $p(t) = (t - \alpha_1)(t - \alpha_2)$ in an algebraic closure).

Hint. Use the equalities $c_1 = -(\alpha_1 + \alpha_2), c_0 = \alpha_1 \alpha_2$.

(b) Let $\Delta(c_0, c_1) := -(4c_1^3 + 27c_0^2)$. Show that for any polynomial $p(t) = t^3 + c_1 t + c_0 \in K[t]$ we have

$$\Delta(c_0, c_1) = \prod_{1 \le i < j \le 3} (\alpha_i - \alpha_j)^2$$

where $\alpha_i \in \bar{K}, 1 \le i \le 3$ are roots of $p(t)$.

(c) Show that there exists a polynomial $\Delta(c_0, c_1, c_2) \in \mathbb{Z}[c_0, c_1, c_2]$ with integer coefficients such that for any field K and a polynomial $p(t) = t^3 + c_2 t^2 + c_1 t + c_0 \in K[t]$, we have

$$\Delta(c_0, c_1, c_2) = \prod_{1 \le i < j \le 3} (\alpha_i - \alpha_j)^2$$

where $\alpha_i \in \bar{K}, 1 \le i \le 3$ are roots of $p(t)$.

Hint. Show the existence of $a \in K$ such that the polynomial $q(u) := p(t + a)$ has a form $q(u) = u^3 + d_1 u + d_0$ for $d_1, d_0 \in K$.

(d) Let $L \supset K, \alpha \in L, p(t) = t^3 + c_2 t^2 + c_1 t + c_0 \in K[t]$ be as in Problem 6.2. Show that if the extension $L : K$ is normal then there exists $\delta \in K$ such that $\Delta(c_0, c_1, c_2) = \delta^2$,

(e) Prove that the converse is also true; that is, if there exists $\delta \in K$ such that $\Delta(c_0, c_1, c_2) = \delta^2$ then the extension $L : K$ is normal.

Hint. Use the results of problem 6.2.

(f) * Assume that there exists $\zeta \in K$ such that $\zeta^3 = 1, \zeta \neq 1$. Let $p(t) = t^3 + c_1 t + c_0 \in K[t]$ be an irreducible polynomial and $\alpha_1, \alpha_2, \alpha_3 \in \bar{K}$ its roots. Let $z := \alpha_1 + \zeta\alpha_2\zeta^2\alpha_3$. Show that

$$(z^3 + \frac{27}{2}c_0)^2 = -\frac{27}{4}\Delta(c_0, c_1, 0)$$

(g) Find formulas for the solution of cubic equations (over fields K such that $\operatorname{char} K \neq 2, 3$).