ON THE PERIODICITY OF COXETER TRANSFORMATIONS AND THE NON-NEGATIVITY OF THEIR EULER FORMS

SEFI LADKANI

Abstract. We show that for piecewise hereditary algebras, the periodicity of the Coxeter transformation implies the non-negativity of the Euler form. Contrary to previous assumptions, the condition of piecewise heredity cannot be omitted, even for triangular algebras, as demonstrated by incidence algebras of posets.

We also give a simple, direct proof, that certain products of reflections, defined for any square matrix A with 2 on its main diagonal, and in particular the Coxeter transformation corresponding to a generalized Cartan matrix, can be expressed as $-A_+^{-1}A_-^t$, where A_+, A_- are closely associated with the upper and lower triangular parts of A.

1. Introduction

Let V be a free abelian group of finite rank and let $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{Z}$ be a non-degenerate $\mathbb{Z}$-bilinear form on V. The *Coxeter transformation* $\Phi : V \to V$ corresponding to $\langle \cdot, \cdot \rangle$ is defined via the equation $\langle x, y \rangle = -\langle y, \Phi x \rangle$ for $x, y \in V$ [14].

The purpose of this paper is to study the relations between positivity properties of the form $\langle \cdot, \cdot \rangle$ and periodicity properties of its Coxeter transformation Φ. Recall that $\langle \cdot, \cdot \rangle$ is *positive* if $\langle x, x \rangle > 0$ for all $0 \neq x \in V$, *non-negative* if $\langle x, x \rangle \geq 0$ for all $x \in V$ and *indefinite* otherwise. The transformation Φ is *periodic* if Φ^m equals the identity I for some integer $m \geq 1$ and *weakly periodic* [20] if $(\Phi^m - I)^n = 0$ for some integers $m, n \geq 1$.

Implications in one direction are given in the paper [20], where linear algebra techniques are used to show that the Coxeter matrix Φ is periodic if $\langle \cdot, \cdot \rangle$ is positive and weakly periodic if $\langle \cdot, \cdot \rangle$ is non-negative. It is much harder to establish implications in the other direction. As already noted in [20], even if Φ is periodic, $\langle \cdot, \cdot \rangle$ may be indefinite, so additional constraints are needed.

An alternative definition of the Coxeter matrix is as a certain product of reflections defined by a generalized Cartan matrix [1, 19], whereas the definition given above is $-C^{-1}C^t$ where C is the matrix of the bilinear form.

We claim similarly to [6], and give a simple, direct proof, that for any square matrix A with 2 on its main diagonal, the product of the n reflections it defines can be expressed as $-A_+^{-1}A_-^t$ where A_+, A_- are closely associated

2000 *Mathematics Subject Classification.* 15A63 ; 16G20 ; 06A11.

Key words and phrases. Coxeter transformation, Euler form, piecewise hereditary, incidence algebras.

with the upper and lower triangular parts of A, see Section 2. This claim can be generalized to products in arbitrary order, and no other conditions on A, such as being generalized Cartan, bipartite [1] or symmetric [12], are needed. In particular, when $\langle \cdot, \cdot \rangle$ is triangular, Φ can be written as a product of the reflections defined by the symmetrization of $\langle \cdot, \cdot \rangle$.

Further connections between periodicity and non-negativity are achieved when we restrict ourselves to pairs $(V, \langle \cdot, \cdot \rangle)$ for which there exists a finite dimensional k-algebra Λ over an algebraically closed field k, having finite global dimension, such that $V \cong K_0(\operatorname{mod} \Lambda)$ and $\langle \cdot, \cdot \rangle$ coincides, under that isomorphism, with the Euler form $\langle \cdot, \cdot \rangle_\Lambda$ of Λ. Here $\operatorname{mod} \Lambda$ denotes the category of finite dimensional right Λ-modules. Since $\operatorname{gl.dim} \Lambda < \infty$, the form $\langle \cdot, \cdot \rangle_\Lambda$ is non-degenerate, hence its Coxeter transformation Φ_Λ is well-defined and coincides with the image in $K_0(\operatorname{mod} \Lambda)$ of the Auslander-Reiten translation on the bounded derived category $\mathcal{D}^b(\operatorname{mod} \Lambda)$.

In Section 3 of the paper, we show that if Λ is *piecewise hereditary*, i.e. its bounded derived category $\mathcal{D}^b(\operatorname{mod} \Lambda)$ is equivalent as a triangulated category to $\mathcal{D}^b(\mathcal{H})$ for a hereditary abelian category $\mathcal{H}$, then the periodicity of Φ_Λ implies the non-negativity of $\langle \cdot, \cdot \rangle_\Lambda$.

In that Section, we also show that when Λ is an incidence algebra of a poset X, the Euler form $\langle \cdot, \cdot \rangle_\Lambda$ and its Coxeter transformation Φ_Λ can be explicitly described in terms of the combinatorics of X.

Previously, [7] claimed that the condition of $\langle \cdot, \cdot \rangle$ being *triangular*, that is, its matrix with respect to some basis of V is upper triangular with ones on the main diagonal, is enough for the periodicity of Φ to imply the non-negativity of $\langle \cdot, \cdot \rangle$. We find however examples of incidence algebras of posets negating this claim, see Section 4.

2. Coxeter transformations of bilinear forms

2.1. **The definition of the Coxeter matrix.** Let V be a free abelian group of finite rank and let $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{Z}$ be a non-degenerate $\mathbb{Z}$-bilinear form on V. Recall that $\langle \cdot, \cdot \rangle$ is *positive* if $\langle v, v \rangle > 0$ for all $0 \neq v \in V$, *non-negative* if $\langle v, v \rangle \geq 0$ for all $v \in V$ and *indefinite* otherwise. The *Coxeter transformation* $\Phi : V \to V$ corresponding to $\langle \cdot, \cdot \rangle$ is defined via the equation $\langle v, w \rangle = - \langle w, \Phi v \rangle$ for all $v, w \in V$ [14].

We consider the elements of $\mathbb{Z}^n$ as column vectors, and denote by M^t the transpose of a matrix M. Let $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbb{Z}^n$. By choosing a $\mathbb{Z}$-basis $v_1, \dots, v_n$ of V, we may identify V with $\mathbb{Z}^n$ and $\langle \cdot, \cdot \rangle$ with the form $\langle \cdot, \cdot \rangle_C$ defined by $\langle x, y \rangle_C = x^t C y$ where $C \in \mathrm{GL}_n(\mathbb{Z})$ is the matrix whose entries are $C_{ij} = \langle v_i, v_j \rangle$ for $1 \leq i, j \leq n$. In other words, $\langle v_i, v_j \rangle = \langle e_i, e_j \rangle_C$. Under this identification, the matrix of Φ is $-C^{-1}C^t$, hence we define the *Coxeter matrix* Φ_C of a matrix $C \in \mathrm{GL}_n(\mathbb{Z})$ to be $\Phi_C = -C^{-1}C^t$.

Note that $v = -C^{-1}C^t v$ if and only if $(C + C^t)v = 0$, hence the geometric multiplicity of the eigenvalue 1 in Φ_C equals the dimension of the radical of the symmetrized bilinear form $C + C^t$.

Definition 2.1. A matrix $\Phi \in \mathrm{GL}_n(\mathbb{Z})$ is *periodic* if $\Phi^m = I$ for some $m \geq 1$. Φ is *weakly periodic* if for some $m \geq 1$, $\Phi^m - I$ is nilpotent.

Definition 2.2. A matrix $C \in \mathrm{GL}_n(\mathbb{Z})$ is *unitriangular* if C is upper triangular and $C_{ii} = 1$ for $1 \leq i \leq n$.

Relations between the positivity of the bilinear form $\langle \cdot, \cdot \rangle_C$ and the periodicity of Φ_C have been studied in [7, 20] and are summarized as follows:

Theorem 2.3. *Let $C \in \mathrm{GL}_n(\mathbb{Z})$. Then:*

(1) [20, (2.8)] *Φ_C is periodic if $\langle \cdot, \cdot \rangle_C$ is positive.*
(2) [20, (3.4)] *Φ_C is weakly periodic if $\langle \cdot, \cdot \rangle_C$ is non-negative.*

However, [20, (3.8)] is an example of a matrix whose Coxeter matrix is periodic but the corresponding bilinear form is indefinite.

2.2. **Alternative definition as a product of reflections.** Following [1, 2, 19], we review an alternative definition of the Coxeter matrix as a product of reflections.

Let A be an $n \times n$ matrix with integer entries satisfying

$$A_{ii} = 2 \qquad 1 \leq i \leq n \tag{A1}$$

$$A_{ij} = 0 \qquad \text{if and only if } A_{ij} = 0, 1 \leq i, j \leq n \tag{A2}$$

The *primitive graph* of A (cf. [2]) is an undirected graph with n vertices, where two vertices $i \neq j$ are connected by an edge if $A_{ij} \neq 0$. The matrix A is *indecomposable* if its primitive graph is connected.

Define *reflections* $r_1, \dots, r_n$ by

$$r_i(e_j) = e_j - A_{ij} e_i \qquad 1 \leq j \leq n \tag{2.1}$$

In other words, r_i is the matrix obtained from the identity matrix by subtracting the i-th row of A. Denote by I the $n \times n$ identity matrix.

Lemma 2.4. *Let A be a matrix satisfying* (A1).

(a) *$r_i^2 = I$ for $1 \leq i \leq n$.*
(b) *If A satisfies also* (A2)*, then $r_i r_j = r_j r_i$ for any two non-adjacent vertices i, j on the primitive graph of A.*

Proof. Since $A_{ii} = 2$, we have $r_i(e_i) = -e_i$, thus

$$r_i^2(e_t) = r_i(e_t - A_{it} e_i) = e_t - A_{it} e_i - A_{it} r_i(e_i) = e_t$$

for all $1 \leq t \leq n$, and the first assertion is proved.

If $A_{ij} = 0$ then $r_i(e_j) = e_j$. The assumptions on A imply that if i, j are not adjacent, then $r_i(e_j) = e_j$ and $r_j(e_i) = e_i$. Therefore, if $1 \leq t \leq n$,

$$r_i r_j(e_t) = r_i(e_t - A_{jt} e_j) = e_t - A_{it} e_i - A_{jt} e_j$$

is symmetric in i and j, hence $r_i r_j = r_j r_i$. ◻

Consider the following two additional properties:

$$A_{ij} \leq 0 \qquad \text{for all } i \neq j \tag{A3}$$

$$\text{The primitive graph of } A \text{ is bipartite} \tag{A4}$$

Definition 2.5. A matrix A is a *generalized Cartan matrix* if it satisfies (A1), (A2) and (A3). A matrix A is *bipartite* if it satisfies (A1), (A2) and (A4).

For a generalized Cartan matrix A and a permutation π of $\{1, 2, \dots, n\}$, a *Coxeter transformation* is defined in [19] by $\Phi(A, \pi) = r_{\pi(1)} r_{\pi(2)} \cdots r_{\pi(n)}$. For a bipartite matrix A, let $\Sigma_1 \amalg \Sigma_2$ be a corresponding partition of $\{1, 2, \dots, n\}$ and consider $R_A = R_1 R_2$ where $R_k = \prod_{i \in \Sigma_k} r_i$, $k = 1, 2$, see [2]. Note that by Lemma 2.4, the matrices R_k do not depend on the order of reflections within each product. Note also that R_A equals $\Phi(A, \pi)$ for a suitable π.

Recall that the *spectrum* of a square matrix Φ with complex entries, denoted $\operatorname{spec}(\Phi)$, is the set of (complex) roots of the characteristic polynomial of Φ. Let $\rho(\Phi) = \max\{|\lambda| : \lambda \in \operatorname{spec}(\Phi)\}$ be the *spectral radius* of Φ.

We recall two results on the spectrum of Coxeter transformations corresponding to generalized Cartan and bipartite matrices.

Theorem 2.6 ([19]). *Let A be an indecomposable generalized Cartan matrix, $\pi \in S_n$. If A is not of finite or affine type, then $\rho(\Phi(A, \pi)) > 1$.*

Theorem 2.7 ([1, p. 63],[2, p. 344]). *Let A be a bipartite matrix.*
(a) $\lambda^2 \in \operatorname{spec}(R_A)$ *if and only if* $\lambda + 2 + \lambda^{-1} \in \operatorname{spec}(A)$.
(b) *If A is also symmetric, then* $\operatorname{spec}(R_A) \subset S^1 \cup \mathbb{R}$.

2.3. **Linking the two definitions.** Let R be any commutative ring with 1 and let $e_1, \dots, e_n$ be a basis of a free R-module of rank n. Let A be an $n \times n$ matrix with entries in R satisfying (A1) (where 2 means $1 + 1$), and define the reflections $r_1, \dots, r_n$ as in (2.1). When we want to stress the dependence of the reflections on A, we shall use the notation $r_1^A, \dots, r_n^A$.

Lemma 2.8. *Let $1 \le s \le n$. Then for every $1 \le t \le n$,*

$$(r_1 \cdots r_s)(e_t) = e_t + \sum_{k=1}^{s} (-1)^k \sum_{1 \le i_1 < i_2 < \dots < i_k \le s} A_{i_1 i_2} \cdots A_{i_{k-1} i_k} A_{i_k t} e_{i_1}$$

Proof. By induction on s, the case $s = 1$ being just the definition of r_1, and for the induction step, expand $r_{s+1}(e_t)$ as $e_t - A_{s+1,t} e_{s+1}$ and use the hypothesis for s.

$$\begin{aligned}
(r_1 \cdots r_s r_{s+1})(e_t) &= (r_1 \cdots r_s)(e_t) - A_{s+1,t}(r_1 \cdots r_s)(e_{s+1}) \\
&= e_t - A_{s+1,t} e_{s+1} + \sum_{k=1}^{s} (-1)^k \sum_{1 \le i_1 < i_2 < \dots < i_k \le s} A_{i_1 i_2} \cdots A_{i_{k-1} i_k} A_{i_k t} e_{i_1} \\
&\quad + \sum_{k=1}^{s} (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \dots < i_k \le s} A_{i_1 i_2} \cdots A_{i_{k-1} i_k} A_{i_k, s+1} A_{s+1,t} e_{i_1} \\
&= e_t - \sum_{k=1}^{s+1} (-1)^k \sum_{1 \le i_1 < i_2 < \dots < i_k \le s+1} A_{i_1 i_2} \cdots A_{i_{k-1} i_k} A_{i_k t} e_{i_1}
\end{aligned}$$

□

Define two $n \times n$ matrices A_+ and A_- by

$$(A_+)_{ij} = \begin{cases} A_{ij} & i < j \\ 1 & i = j \\ 0 & i > j \end{cases} \qquad (A_-)_{ij} = \begin{cases} A_{ji} & i < j \\ 1 & i = j \\ 0 & i > j \end{cases}$$

Then $A = A_+ + A_-^t$, and one can think of A_+, A_- as the upper and lower triangular parts of A. The matrices A_+ and A_- are invertible since $A_+ - I$ and $A_- - I$ are nilpotent. Note that A is symmetric if and only if $A_+ = A_-$

Theorem 2.9. *If A satisfies* (A1)*, then $r_1^A r_2^A \cdots r_n^A = -A_+^{-1} A_-^t$.*

Proof. By Lemma 2.8 with $s = n$,

$$(r_1 \cdots r_n)(e_t) = e_t + \sum_{k=1}^{n} (-1)^k \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} A_{i_1 i_2} \cdots A_{i_{k-1} i_k} A_{i_k t} e_{i_1}$$

This can be written in matrix form, using the definition of A_+, as follows:

$$\begin{aligned} r_1 \cdots r_n &= I + \sum_{k=1}^{n} (-1)^k (A_+ - I)^{k-1} A \\ &= I - (I - (A_+ - I) + (A_+ - I)^2 - \dots) A \\ &= I - A_+^{-1}(A_+ + A_-^t) = -A_+^{-1} A_-^t \end{aligned}$$

□

Remark 2.10. Theorem 2.9 is still true when we drop the condition (A1) and slightly change the definition of A_-, by $(A_-)_{ii} = A_{ii} - 1$ for $1 \le i \le n$. However, in that case the matrices r_i are no longer reflections.

Theorem 2.9 provides a link between the definition of the Coxeter matrix as a specific automorphism of the bilinear form and its definition as a product of n reflections, as shown by the following corollary.

Corollary 2.11. *Let $C \in \mathrm{GL}_n(\mathbb{Z})$ be a unitriangular matrix. Then $\Phi_C = \Phi(C + C^t, id)$, that is, $\Phi_C = r_1^A r_2^A \cdots r_n^A$ for $A = C + C^t$.*

In fact, this corollary is proved in [12] for the case where Φ_C is a Coxeter element in an arbitrary Coxeter group of finite rank represented as a group of linear transformations on a real inner product space, so that the Cartan matrix A is symmetric.

Proof. Apply Theorem 2.9 for the matrix $A = C + C^t$, which satisfies (A1), (A2) and $A_+ = A_- = C$. □

Denote by S_n the group of permutations on $\{1, 2, \dots, n\}$ and let $\pi \in S_n$. One could deduce a generalized version of Theorem 2.9 for the product of the n reflections in an arbitrary order by proving an analogue of Lemma 2.8 for arbitrary π. Instead, we will derive the generalized version from the original one using permutation matrices.

Define the permutation matrix P_π by $P_\pi(e_i) = e_{\pi(i)}$ for all $1 \le i \le n$. Note that $P_\pi^{-1} = P_\pi^t$. Given a matrix A, let A_π denote the matrix $P_\pi^{-1} A P_\pi$, so that $(A_\pi)_{ij} = A_{\pi(i)\pi(j)}$. Obviously, if A satisfies (A1), so does A_π.

Lemma 2.12. *Let $1 \le i \le n$. Then $r_i^{A_\pi} = P_\pi^{-1} r_{\pi(i)}^A P_\pi$.*

Proof. For all $1 \le t \le n$,

$$\left(P_\pi^{-1} r_{\pi(i)}^A P_\pi \right)(e_t) = P_\pi^{-1} \left(e_{\pi(t)} - A_{\pi(i)\pi(t)} e_{\pi(i)} \right) = e_t - A_{\pi(i)\pi(t)} e_i = r_i^{A_\pi}(e_t)$$

□

Define two $n \times n$ matrices $A_{\pi,+}$ and $A_{\pi,-}$ by

$$(A_{\pi,+})_{ij} = \begin{cases} A_{ij} & \pi^{-1}(i) < \pi^{-1}(j) \\ 1 & i = j \\ 0 & \text{otherwise} \end{cases} \qquad (A_{\pi,-})_{ij} = \begin{cases} A_{ji} & \pi^{-1}(i) < \pi^{-1}(j) \\ 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

Direct calculation shows that $A_{\pi,+} = P_\pi (A_\pi)_+ P_\pi^{-1}$, $A_{\pi,-} = P_\pi (A_\pi)_- P_\pi^t$ and $A = A_{\pi,+} + A_{\pi,-}^t$.

Corollary 2.13. *Let A satisfy* (A1) *and let $\pi \in S_n$. Then*

$$r^A_{\pi(1)} r^A_{\pi(2)} \dots r^A_{\pi(n)} = -A_{\pi,+}^{-1} A_{\pi,-}^t$$

Proof. By Lemma 2.12 and Theorem 2.9 applied for A_π,

$$\begin{aligned} r^A_{\pi(1)} r^A_{\pi(2)} \dots r^A_{\pi(n)} &= P_\pi \left(r_1^{A_\pi} r_2^{A_\pi} \dots r_n^{A_\pi} \right) P_\pi^{-1} = -P_\pi (A_\pi)_+^{-1} (A_\pi)_-^t P_\pi^t \\ &= -\left(P_\pi (A_\pi)_+^{-1} P_\pi^{-1} \right) \left(P_\pi (A_\pi)_-^t P_\pi^t \right) = -A_{\pi,+}^{-1} A_{\pi,-}^t \end{aligned}$$

□

3. Periodicity and non-negativity for piecewise hereditary algebras and posets

Let k be a field, and let $\mathcal{A}$ be an abelian k-category of finite global dimension with finite dimensional Ext-spaces. Denote by $\mathcal{D}^b(\mathcal{A})$ its bounded derived category and by $K_0(\mathcal{A})$ its Grothendieck group. The expression

$$\langle X, Y \rangle_{\mathcal{A}} = \sum_{i \in \mathbb{Z}} (-1)^i \dim_k \operatorname{Hom}_{\mathcal{D}^b(\mathcal{A})}(X, Y[i])$$

is well-defined for $X, Y \in \mathcal{D}^b(\mathcal{A})$ and induces a $\mathbb{Z}$-bilinear form on $K_0(\mathcal{A})$, known as the *Euler form*. When $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ is non-degenerate, the unique transformation $\Phi_{\mathcal{A}} : K_0(\mathcal{A}) \to K_0(\mathcal{A})$ satisfying $\langle x, y \rangle_{\mathcal{A}} = -\langle y, \Phi_{\mathcal{A}} x \rangle_{\mathcal{A}}$ for all $x, y \in K_0(\mathcal{A})$ is called the *Coxeter transformation* of $\mathcal{A}$. For more details we refer the reader to [15].

Two such abelian k-categories $\mathcal{A}$ and $\mathcal{B}$ are said to be *derived equivalent* if there exists a triangulated equivalence $F : \mathcal{D}^b(\mathcal{A}) \simeq \mathcal{D}^b(\mathcal{B})$. In this case, the forms $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{B}}$ are equivalent over $\mathbb{Z}$, hence the positivity properties of the Euler form and the periodicity properties of the Coxeter transformation are invariants of derived equivalence.

Let Λ be a finite dimensional algebra of finite global dimension over an algebraically closed field k, and consider the k-category $\operatorname{mod} \Lambda$ of finitely generated right modules over Λ. Denote by $\mathcal{D}^b(\Lambda)$ its bounded derived category, by $K_0(\Lambda)$ its Grothendieck group and by $\langle \cdot, \cdot \rangle_\Lambda$ the Euler form. Then $K_0(\Lambda)$ is free of finite rank, with a $\mathbb{Z}$-basis consisting of the representatives of the isomorphism classes of simple modules in $\operatorname{mod} \Lambda$. The form $\langle \cdot, \cdot \rangle_\Lambda$ is non-degenerate, and its Coxeter transformation Φ_Λ coincides with the linear map on $K_0(\Lambda)$ induced by the Auslander-Reiten translation on $\mathcal{D}^b(\Lambda)$. For more details see [9, (III.1)], [18, (2.4)] or [15].

3.1. **Path algebras of quivers without oriented cycles.** The first example of algebras Λ for which the connection between the positivity of $\langle \cdot, \cdot \rangle_\Lambda$ and the periodicity of Φ_Λ is completely understood is the class of path algebras of quivers without oriented cycles, or more generally hereditary algebras, see [15, Theorem 18.5]. We briefly review the main results.

A (finite) *quiver* Q is a directed graph with a finite number of vertices and edges. The *underlying graph* of Q is the undirected graph obtained from Q by forgetting the orientations of the edges. An *oriented cycle* is a nontrivial path in Q starting and ending at the same vertex. The *path algebra* kQ is the algebra over k having as a k-basis the set of all (oriented) paths in Q; the product of two paths is their composition, if defined, and zero otherwise.

When Q has no oriented cycles, the path algebra kQ is hereditary and finite-dimensional. Denote by $\langle \cdot, \cdot \rangle_Q$ its Euler form and by Φ_Q its Coxeter transformation. The matrix of $\langle \cdot, \cdot \rangle_Q$ with respect to the basis of simple modules is unitriangular, and its symmetrization is generalized Cartan. The relations between the periodicity of Φ_Q and the positivity of $\langle \cdot, \cdot \rangle_Q$ are summarized in the following well-known proposition, see [1, 3, 5, 19] and [18, (1.2)].

Proposition 3.1. *Let Q be a connected quiver without oriented cycles. Then:*

(a) Φ_Q *is periodic if and only if* $\langle \cdot, \cdot \rangle_Q$ *is positive, equivalently the underlying graph of Q is a Dynkin diagram of type A, D or E.*
(b) Φ_Q *is weakly periodic if and only if* $\langle \cdot, \cdot \rangle_Q$ *is non-negative, equivalently the underlying graph of Q is a Dynkin diagram or an extended Dynkin diagram of type $\tilde{A}$, $\tilde{D}$ or $\tilde{E}$.*

3.2. **Canonical algebras.** Another interesting class of algebras for which the connection between non-negativity and periodicity is established are the canonical algebras, introduced in [18].

The Grothendieck group and the Euler form of canonical algebras were thoroughly studied in [14]. If Λ is canonical of type $(\mathbf{p}, \boldsymbol{\lambda})$ where $\mathbf{p} = (p_1, \dots, p_t)$ and $\boldsymbol{\lambda} = (\lambda_3, \dots, \lambda_t)$ is a sequence of pairwise distinct elements of $k \setminus \{0\}$, then the rank of $K_0(\Lambda)$ is $\sum_{i=1}^t p_i - (t-2)$ and the characteristic polynomial of the Coxeter transformation Φ_Λ equals $(T-1)^2 \prod_{i=1}^t \frac{T^{p_i}-1}{T-1}$ [14, Prop. 7.8)]. In particular, $\rho(\Phi_\Lambda) = 1$ and the eigenvalues of Φ_Λ are roots of unity, hence Φ_Λ is weakly periodic.

The following proposition follows from [14, Prop. 10.3], see also [16].

Proposition 3.2. *Let Λ be a canonical algebra of type* $(\mathbf{p}, \boldsymbol{\lambda})$*. If Φ_Λ is periodic then* $\mathbf{p}$ *is one of* $(2,3,6)$*,* $(2,4,4)$*,* $(3,3,3)$ *or* $(2,2,2,2)$*. In any of these cases,* $\langle \cdot, \cdot \rangle_\Lambda$ *is non-negative.*

3.3. **Extending to piecewise hereditary algebras.** We extend the results of the previous sections to the class of all piecewise hereditary algebras.

Definition 3.3. An algebra Λ over k is *piecewise hereditary* if there exist a hereditary abelian category $\mathcal{H}$ and a triangulated equivalence $\mathcal{D}^b(\Lambda) \simeq \mathcal{D}^b(\mathcal{H})$.

Theorem 3.4. *Let k be an algebraically closed field and let Λ be a finite dimensional piecewise hereditary k-algebra. If Φ_Λ is periodic, then $\langle \cdot, \cdot \rangle_\Lambda$ is non-negative.*

Proof. By definition, there exists a hereditary category $\mathcal{H}$ and an equivalence of triangulated categories $F : \mathcal{D}^b(\Lambda) \simeq \mathcal{D}^b(\mathcal{H})$. By the invariance under derived equivalence, it is enough to prove the theorem for $\Phi_\mathcal{H}$ and $\langle \cdot, \cdot \rangle_\mathcal{H}$. Moreover, we can assume that $\mathcal{H}$ is connected.

Now $\mathcal{H}$ is an Ext-finite k-category and $F(\Lambda_\Lambda)$ is a tilting complex in $\mathcal{D}^b(\mathcal{H})$, so by [11, Theorem 1.7], $\mathcal{H}$ admits a *tilting object*, that is, an object T with $\mathrm{Ext}^1_\mathcal{H}(T, T) = 0$ such that for any object X of $\mathcal{H}$, the condition $\mathrm{Hom}_\mathcal{H}(T, X) = 0 = \mathrm{Ext}^1_\mathcal{H}(T, X)$ implies that $X = 0$.

By the classification of hereditary connected Ext-finite k-categories with tilting object up to derived equivalence over an algebraically closed field [10], $\mathcal{H}$ is derived equivalent to $\mathrm{mod}\, H$ for a finite dimensional hereditary algebra H or to $\mathrm{mod}\, \Lambda$ for a canonical algebra Λ. Again by invariance under derived equivalence we may assume that $\mathcal{H} = \mathrm{mod}\, H$ or $\mathcal{H} = \mathrm{mod}\, \Lambda$.

For $\mathcal{H} = \mathrm{mod}\, H$, we can replace H by a path algebra of a finite connected quiver without oriented cycles, and then use Proposition 3.1. For $\mathcal{H} = \mathrm{mod}\, \Lambda$, the result follows from Proposition 3.2. □

3.4. **Incidence algebras of posets.** Let X be a finite partially ordered set (*poset*) and let k be a field. The *incidence algebra* kX is the k-algebra spanned by elements e_{xy} for the pairs $x \le y$ in X, with multiplication defined by $e_{xy} e_{zw} = \delta_{yz} e_{xw}$. Finite dimensional right modules over kX can be identified with commutative diagrams of finite dimensional k-vector spaces over the *Hasse diagram* of X which is the directed graph whose vertices are the points of X, with an arrow from x to y if $x < y$ and there is no $z \in X$ with $x < z < y$.

We recollect the basic facts on the Euler form of posets and refer the reader to [13] for details. The algebra kX is of finite global dimension, hence its Euler form, denoted $\langle \cdot, \cdot \rangle_X$, is well-defined and non-degenerate. Denote by C_X, Φ_X the matrices of $\langle \cdot, \cdot \rangle_X$ and its Coxeter transformation with respect to the basis of simple kX-modules.

The *incidence matrix* of X, denoted $\mathbf{1}_X$, is the $X \times X$ matrix defined by

$$(\mathbf{1}_X)_{xy} = \begin{cases} 1 & x \le y \\ 0 & \text{otherwise} \end{cases}$$

By extending the partial order on X to a linear order, we can always arrange the elements of X such that the incidence matrix is unitriangular. In particular, $\mathbf{1}_X$ is invertible over $\mathbb{Z}$. Recall that the *Möbius function* $\mu_X : X \times X \to \mathbb{Z}$ is defined by $\mu_X(x, y) = (\mathbf{1}_X^{-1})_{xy}$.

Lemma 3.5 ([13, Prop. 3.11])**.** $C_X = \mathbf{1}_X^{-1}$.

Lemma 3.6. *Let $x, y \in X$. Then $(\Phi_X)_{xy} = -\sum_{z\,:\,z \ge x} \mu_X(y, z)$.*

Proof. Since $\Phi_X = -C_X^{-1} C_X^t = -\mathbf{1}_X \mathbf{1}_X^{-t}$,

$$(\Phi_X)_{xy} = -\sum_{z \in X} (\mathbf{1}_X)_{xz} (\mathbf{1}_X^{-1})_{yz} = -\sum_{z\,:\,z \ge x} \mu_X(y, z)$$

□

When the Hasse diagram of X has the property that any two vertices x, y are connected by at most one directed path, the Möbius function takes a very simple form, namely

$$\mu_X(x,y) = \begin{cases} 1 & y = x \\ -1 & x \to y \text{ is an edge in the Hasse diagram} \\ 0 & \text{otherwise} \end{cases}$$

In this case, Lemma 3.6 coincides with Proposition 3.1 of [4], taking the Hasse diagram as the quiver.

Lemma 3.7. *If X and Y are posets, then*

$$C_{X\times Y} = C_X \otimes C_Y \qquad \Phi_{X\times Y} = -\Phi_X \otimes \Phi_Y$$

Proof. Observe that $\mathbf{1}_{X\times Y} = \mathbf{1}_X \otimes \mathbf{1}_Y$. □

Corollary 3.8. *Let X, Y be posets with periodic Coxeter matrices. Then $X \times Y$ has also periodic Coxeter matrix.*

Since non-negativity of forms is not preserved under tensor products, Corollary 3.8 can be used to construct posets with periodic Coxeter matrix but with indefinite Euler form, see Example 4.4.

4. Examples

For a poset X, let C_X, Φ_X be as in the previous section. In particular we may assume that C_X is unitriangular. The symmetrization $A_X = C_X + C_X^t$ satisfies (A1) and (A2), but in general it is not bipartite nor generalized Cartan.

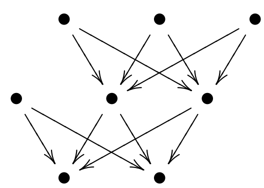
4.1. **Spectral properties of Φ_X.**

Example 4.1. *The spectrum of Φ_X does not determine that of A_X (Compare with Theorem 2.7a).*

The four posets whose Hasse diagrams are depicted in Figure 1 are derived equivalent (as they are all piecewise hereditary of type D_5), hence their Coxeter matrices are similar and have the same spectrum, namely the roots of the characteristic polynomial $x^5 + x^4 + x + 1$. However, the spectra of the corresponding symmetrized forms are different. Figure 1 also shows for each poset X the characteristic polynomial of the matrix of its symmetrized form.

Example 4.2. *A poset X with* $\operatorname{spec}\Phi_X \not\subseteq S^1 \cup \mathbb{R}$ *(Compare with Theorem 2.7b).*

Let X be the following poset.



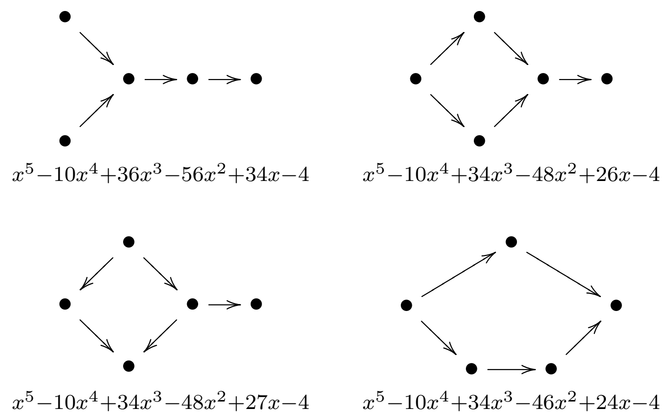


FIGURE 1. Derived equivalent posets with different spectra of the corresponding symmetrized bilinear forms.

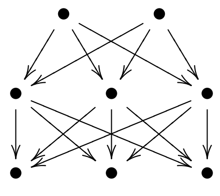
The characteristic polynomial of Φ_X is $(x+1)^4(x^4-2x^3+6x^2-2x+1)$, whose roots, besides -1, are $z, \bar{z}, z^{-1}, \bar{z}^{-1}$ with $\Re z = \frac{1+\sqrt{2\sqrt{3}-3}}{2}$ and $|z|^2 = \frac{1+\sqrt{2\sqrt{3}-3}}{2-\sqrt{3}} - 1$. These four roots are neither real nor on the unit circle.

An example of similar spectral behavior for path algebra of a quiver is given in [15, Example 18.1].

Note that for all posets X with 7 elements or less, $\operatorname{spec}(\Phi_X) \subseteq S^1 \cup \mathbb{R}$. This was verified using the database [8] and the MAGMA software package.

4.2. **Counterexamples to** [7, Prop. 1.2]**.** We give two examples of posets showing that in general, for triangular algebras, the periodicity of the Coxeter transformation (and even of the Auslander-Reiten translation up to a shift) does not imply the non-negativity of the Euler form.

Example 4.3. Consider the poset X with the following Hasse diagram.



Then $\Phi_X^6 = I$ but $v^t C_X v = -1$ for $v = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}^t$ (the vertices are ordered in layers from top to bottom).

Example 4.4. Let $X = A_3 \times D_4$ with the following orientations:

$$\begin{tikzcd} 1 \arrow[dr] & & 2 \arrow[dl] \\ & 3 & \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[dr] & & 2 \arrow[dl] \\ & 3 \arrow[d] & \\ & 4 & \end{tikzcd}$$

The Hasse diagram of X is given by

$$\begin{tikzcd}
1,1 \arrow[dr] \arrow[drr] & & 1,2 \arrow[dl] \arrow[drr] & & 2,1 \arrow[dll] \arrow[dr] & & 2,2 \arrow[dll] \arrow[dl] \\
& 1,3 \arrow[d] \arrow[drr] & 3,1 \arrow[dr] & & 3,2 \arrow[dl] & 2,3 \arrow[dll] \arrow[d] & \\
& 1,4 \arrow[drr] & & 3,3 \arrow[d] & & 2,4 \arrow[dll] & \\
& & & 3,4 & & &
\end{tikzcd}$$

so that X contains the following wild quiver as a subposet.

$$\begin{tikzcd}
1,3 \arrow[drr] & 3,1 \arrow[dr] & & 3,2 \arrow[dl] & 2,3 \arrow[dll] \\
& & 3,3 \arrow[d] & & \\
& & 3,4 & &
\end{tikzcd}$$

It follows [17] that kX is not of finite representation type, hence by [21, Theorem 6] the form $\langle \cdot, \cdot \rangle_X$ is not weakly positive, that is, there exists a vector $v \neq 0$ with non-negative coordinates such that $\langle v, v \rangle_X \leq 0$.

Moreover, we can exhibit a non-negative vector v such that $v^t C_X v = -1$, namely $v = (v_x)_{x \in X}$ where the integers v_x are placed at the vertices as in the following picture:

$$\begin{tikzcd}
0 \arrow[dr] \arrow[drr] & & 0 \arrow[dl] \arrow[drr] & & 0 \arrow[dll] \arrow[dr] & & 0 \arrow[dll] \arrow[dl] \\
& 1 \arrow[d] \arrow[drr] & 1 \arrow[dr] & & 1 \arrow[dl] & 1 \arrow[dll] \arrow[d] & \\
& 1 \arrow[drr] & & 2 \arrow[d] & & 1 \arrow[dll] & \\
& & & 1 & & &
\end{tikzcd}$$

On the other hand, the Coxeter matrices of the quivers A_3 and D_4 are periodic, their orders are 4 and 6 respectively. By Corollary 3.8, the Coxeter matrix of X is periodic of order 12.

Contrary to Example 4.3, one can show that not only the image Φ_X of the Auslander-Reiten translation $\tau_X : \mathcal{D}^b(X) \to \mathcal{D}^b(X)$ in the Grothendieck group is periodic, but also that actually $\tau_X^e \simeq [d]$ for some integers $d, e \geq 1$.

References

1. Norbert A'Campo, *Sur les valeurs propres de la transformation de Coxeter*, Invent. Math. **33** (1976), no. 1, 61–67.
2. S. Berman, Y. S. Lee, and R. V. Moody, *The spectrum of a Coxeter transformation, affine Coxeter transformations, and the defect map*, J. Algebra **121** (1989), no. 2, 339–357.
3. I.N. Bernstein, I.M. Gel'fand, and V.A. Ponomarev, *Coxeter functors and Gabriel's theorem.*, Russ. Math. Surv. **28** (1973), no. 2, 17–32 (English).
4. Axel Boldt, *Methods to determine Coxeter polynomials*, Linear Algebra Appl. **230** (1995), 151–164.
5. Nicolas Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2002, Translated from the 1968 French original by Andrew Pressley.

6. A. J. Coleman, *Killing and the Coxeter transformation of Kac-Moody algebras*, Invent. Math. **95** (1989), no. 3, 447–477.
7. José A. de la Peña, *Periodic Coxeter matrices*, Linear Algebra Appl. **365** (2003), 135–142, Special issue on linear algebra methods in representation theory.
8. A. Cheryl Gann and A. Robert Proctor, *Chapel Hill Poset Atlas*, `http://www.unc.edu/~rap/Posets`.
9. Dieter Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, vol. 119, Cambridge University Press, Cambridge, 1988.
10. ———, *A characterization of hereditary categories with tilting object*, Invent. Math. **144** (2001), no. 2, 381–398.
11. Dieter Happel and Idun Reiten, *Directing objects in hereditary categories*, Trends in the representation theory of finite-dimensional algebras (Seattle, WA, 1997), Contemp. Math., vol. 229, Amer. Math. Soc., Providence, RI, 1998, pp. 169–179.
12. Robert B. Howlett, *Coxeter groups and M-matrices*, Bull. London Math. Soc. **14** (1982), no. 2, 137–141.
13. Sefi Ladkani, *On derived equivalences of categories of sheaves over finite posets*, J. Pure Appl. Algebra (2007), doi:10.1016/j.jpaa.2007.06.005.
14. Helmut Lenzing, *A K-theoretic study of canonical algebras*, Representation theory of algebras (Cocoyoc, 1994), CMS Conf. Proc., vol. 18, Amer. Math. Soc., Providence, RI, 1996, pp. 433–454.
15. ———, *Coxeter transformations associated with finite-dimensional algebras*, Computational methods for representations of groups and algebras (Essen, 1997), Progr. Math., vol. 173, Birkhäuser, Basel, 1999, pp. 287–308.
16. Helmut Lenzing and José Antonio de la Peña, *Wild canonical algebras*, Math. Z. **224** (1997), no. 3, 403–425.
17. Michèle Loupias, *Indecomposable representations of finite ordered sets*, Representations of algebras (Proc. Internat. Conf., Carleton Univ., Ottawa, Ont., 1974), Springer, Berlin, 1975, pp. 201–209. Lecture Notes in Math., Vol. 488.
18. Claus Michael Ringel, *Tame algebras and integral quadratic forms*, Lecture Notes in Mathematics, vol. 1099, Springer-Verlag, Berlin, 1984.
19. ———, *The spectral radius of the Coxeter transformations for a generalized Cartan matrix*, Math. Ann. **300** (1994), no. 2, 331–339.
20. Masahisa Sato, *Periodic Coxeter matrices and their associated quadratic forms*, Linear Algebra Appl. **406** (2005), 99–108.
21. Sergey Yuzvinsky, *Linear representations of posets, their cohomology and a bilinear form*, European J. Combin. **2** (1981), no. 4, 385–397.

Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem 91904, Israel

E-mail address: `sefil@math.huji.ac.il`