joint w/ Y. Bae, D. Holmes, R. Pandharipande, R. Schwarz

§1 Two compactifications of loci of k-differentials

Let $g, n, k \geq 0$, $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ w/ $\sum a_i = k \cdot (2g-2+n)$.

$$\mathcal{H}_g^k(A) = \{ (C, p_1, \ldots, p_n) \mid \omega_C^{\otimes k} \cong \mathcal{O}(\textstyle\sum a_i p_i) \} \subseteq \mathcal{M}_{g,n}$$

$\Leftrightarrow$ $\exists$ merom. k-diff. η on C with $\operatorname{div}(\eta) = \sum a_i p_i$

Q · How to compactify in $\overline{\mathcal{M}}_{g,n}$?
· Natural cycle class in $CH^*(\overline{\mathcal{M}}_{g,n})$?

A1 Closure $\overline{\mathcal{H}}_g(A) \subseteq \overline{\mathcal{M}}_{g,n}$ $\rightsquigarrow$ strata of k-differentials
→ minimal compactification

→ [Bainbridge-Chen-Gendron-Grushevsky-Möller '16]
Characterization of $(C, p_1, \ldots, p_n) \in \overline{\mathcal{H}}_g(A)$
$\rightsquigarrow$ $\exists$ meromorph. k-diff. on comp. of C, poles & zeros at nodes, k-residue conditions

→ [BCGGM '19, Costantini-Möller-Zachhuber '19]
Construct smooth, modular compactification

$$\mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(A) \longrightarrow \overline{\mathcal{H}}_g^k(A) \subseteq \overline{\mathcal{M}}_{g,n}$$

→ [Janda-Pandharipande-Pixton-Zvonkine '19] "Tautolog. relat. via r-spin structures"
$k=1$, $A \geq 0$: Conjectural relation

Witten's r-spin class

$$[\overline{\mathcal{H}}_g^1(A)] = (-1)^g \underbrace{r^{g-1} W_{g,n}^r(a_1, \ldots, a_n)}_{\text{poly. in } r \text{ for } r \gg 0}\Big|_{r=0} \in H^{2(g-1)}(\overline{\mathcal{M}}_{g,n})$$

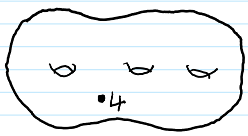
$\Downarrow$

<u>Generalization</u>

residues at all poles vanish

$$[\overline{\mathcal{H}}_g^1(-a_1, \ldots, -a_\ell, a_{\ell+1}, \ldots, a_n)^{\text{res}=0}] = (-1)^g r^{g-1+\ell} W_{g,n}^r(r-a_1, \ldots, r-a_\ell, a_{\ell+1}, \ldots, a_n)\Big|_{r=0} \in H^{2(g-1+\ell)}(\overline{\mathcal{M}}_{g,n})$$

<u>Idea of Pf. of $\Downarrow$</u>



boundary $\rightsquigarrow$ pullback

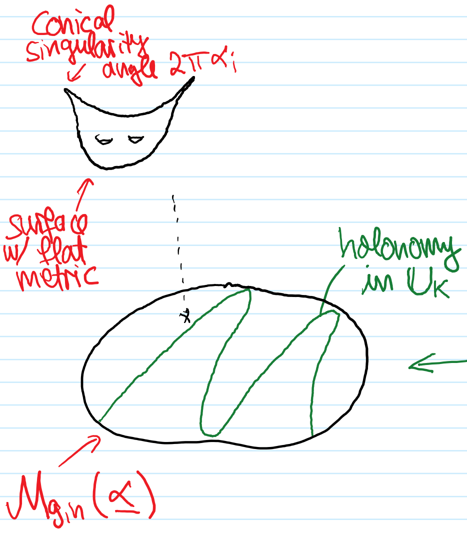
$$[\overline{\mathcal{H}}_3^1(4)] \xrightarrow{\text{[BCGGM]}} {}^{0}\!\!\times [\overline{\mathcal{H}}_1^1(0)] \otimes [\overline{\mathcal{H}}_1^1(-2,-2,4)^{\text{res}=0}] \otimes [\overline{\mathcal{H}}_1^1(0)]^{\neq 0}$$

$$\| \qquad\qquad\qquad ((\qquad \| \qquad\qquad \|! \qquad\qquad \|$$

$$(-1)^3 r^{3-1} W_{3,1}^r(4)\Big|_{r=0} \xrightarrow{\text{CohFT}} \left[(-1)^1 r^0 W_{1,1}^r(0)\right] \otimes \left[(-1)^1 r^2 W_{1,3}^r(r-2, r-2, 4)\right] \otimes \left[(-1)^1 r^0 W_{1,1}^r(0)\right]\Big|_{r=0}$$

→ [Sauvaget '20]

Volumes of moduli spaces of flat surfaces $\longleftrightarrow$ Intersect. numb. of $[\overline{\mathcal{H}}_g^K(A)]$ w/ Tautol. classes



$\alpha_i \in \mathbb{Q}$, K suff. divisible

$\Downarrow$

$\longleftarrow \mathcal{H}_g^K(K\cdot\underline{\alpha})$

↑ equidistributes as $K \to \infty$

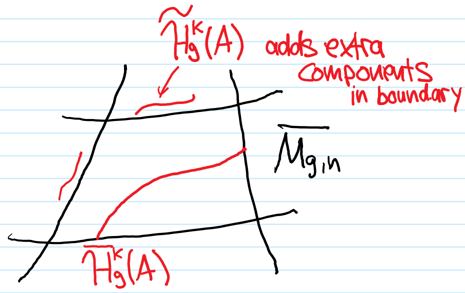
<u>Problem</u> (Still) no nice formula for $[\overline{\mathcal{H}}_g^K(A)]$ in general

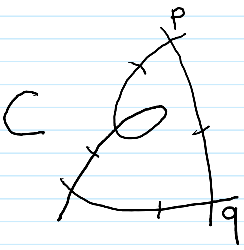
<u>A2</u> Moduli $\widetilde{\mathcal{H}}_g^K(A) \subseteq \overline{M}_{g,n}$ of twisted K-differentials [Farkas-Pandharipande '15]

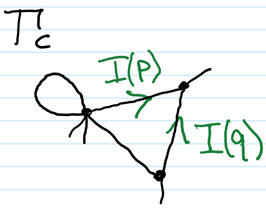
$$\widetilde{\mathcal{H}}_g^K(A) = \{(C, p_1, \ldots, p_n) \mid (*)\} \subseteq \overline{M}_{g,n},$$

closed

$$\widetilde{\mathcal{H}}_g^K(A) \cap M_{g,n} = \mathcal{H}_g^K(A)$$





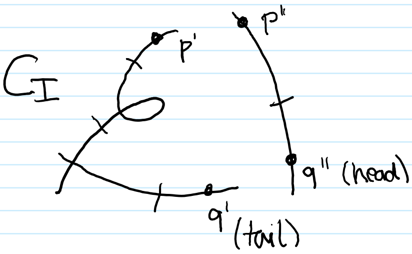


twist I

$I(p), I(q) \in \mathbb{Z}_{>0}$

↳ no <u>strict</u> cycles

$\uparrow \nu_I$ normalizing twisted nodes N_I



$(*)$: $\exists$ twist I on Γ_C such that

$$\nu_I^* \omega_C^{\otimes K} \cong \nu_I^* \mathcal{O}_C\left(\sum a_i p_i\right) \otimes \mathcal{O}_{C_I}\left(\sum_{q \in N_I} I(q)(q'' - q')\right)$$

$$\Leftrightarrow \quad \omega_{C_I}^{\otimes K} \cong \mathcal{O}_{C_I}\left(\sum a_i p_i + \sum_{q \in N_I} (-I(q)-K)q' + (I(q)-K)q''\right)$$

$\rightsquigarrow$ Motivation for definition?

line bdle on C
total deg. 0

$(C, p_1, \dots, p_n) \mapsto (C, \omega_C^{\otimes k}(-\sum a_i p_i))$

$$\begin{tikzcd} & \mathrm{Pic}_{g,0} = \{(C,\mathcal{L})\} \arrow[d, "\pi"'] \\ \overline{M}_{g,n} \arrow[ur, "\sigma_{A,k}"] \arrow[r] & \mathfrak{M}_g = \{C\} \arrow[u, bend right, "e"'] \end{tikzcd} \qquad \mathcal{L} = \mathcal{O}_C$$

prestable genus g

Prop (BHPSS'20)
$\widetilde{H}_g^k(A) = \sigma_{A,k}^{-1}(\overline{e})$

closure of image of e in $\mathrm{Pic}_{g,0}$

$\rightsquigarrow$ $\overline{e} \subset \mathrm{Pic}_{g,0}$ has pure codim g; what about $\widetilde{H}_g^k(A)$?

§2 Dimension theory & weighted fundamental class

Thm (F-P'15 $(k=1)$, S'16 $(k>1)$)
For $k \geq 1$, $\widetilde{H}_g^k(A)$ has pure codim g in $\overline{M}_{g,n}$,
except if $A = k \cdot A'$ for $A' \in \mathbb{Z}_{\geq 0}^n$, in which case

$$\widetilde{H}_g^1(A') \subseteq \widetilde{H}_g^k(A)$$

is a union of comp. of codim $g-1$.

Idea of Pf: over $M_{g,n}$: $\sigma_{A,k}$ and e meet transversally (Deformation theory)
in $\partial \overline{M}_{g,n}$: recursive argument (see below) $\square$

$\rightsquigarrow$ What about cycle theory?

Conjecture A (Janda-Pandharipande-Pixton-Zvonkine '15 $(k=1)$, S '16 $(k>1)$)

Let $k \geq 1$ and $A \neq kA'$ for $A' \in \mathbb{Z}_{\geq 0}^n$. $\rightsquigarrow$ $\widetilde{H}_g^k(A)$ pure codim g
Then

$$\sum_{\substack{Z \text{ component} \\ \text{of } \widetilde{H}_g^k(A)}} m_Z \cdot [Z] = 2^{-g} P_g^{g,k}(\widetilde{A}) \in CH^g(\overline{M}_{g,n}) \qquad (\star)$$

m_Z: explicit pos. integer

$\widetilde{A} = (a_1+k, \dots, a_n+k)$

$P_g^{g,k}$: Pixton's formula for generalized double ramification cycle $\in R^g(\overline{M}_{g,n})$

Proof ([HS'19, BHPSS'20])

Recall

$$\left.\begin{tikzcd} & \mathrm{Pic}_{g,0} \arrow[d, "\pi"'] \\ \overline{M}_{g,n} \arrow[ur, "\sigma_{A,K}"] \arrow[r] & \mathfrak{M}_g \arrow[u, bend right, "e"'] \end{tikzcd}\right\} \quad \widetilde{H}^K_g(A) = \sigma_{A,K}^{-1}(\overline{e})$$

$\rightarrow$ [HS'19] $\begin{pmatrix}\text{Intersection multiplicity} \\ \text{of } \sigma_{A,K} \text{ and } \overline{e} \text{ along} \\ Z \subset \widetilde{H}^K_g(A)\end{pmatrix} = m_Z \Rightarrow$ LHS of $(\ast) = \sigma_{A,K}^{\ast}([\overline{e}])$

$\|$

$\rightarrow$ [BHPSS'20] Show Pixton style formula $[\overline{e}] = P^g_g \in CH^g(\mathrm{Pic}_{g,0})$ RHS of $(\ast) = \sigma^{\ast}_{A,K}(P^g_g)$

short computation

$\square$

$\Rightarrow [\overline{e}] \in CH^g(\mathrm{Pic}_{g,0})$ is the <u>universal twisted DR cycle</u>

$\rightsquigarrow$ all classical DR cycles are pullbacks under $\sigma_{A,K} : \overline{M}_{g,n} \to \mathrm{Pic}_{g,0}$

$\rightsquigarrow$ What about cycles $[\widetilde{H}^K_g(A)]$?

<u>Conj. A</u> $[\widetilde{H}^K_g(A)] + \begin{pmatrix}\text{boundary comp.} \\ \text{of } \widetilde{H}^K_g(A)\end{pmatrix} = \begin{pmatrix}\text{explicit} \\ \text{formula}\end{pmatrix}$

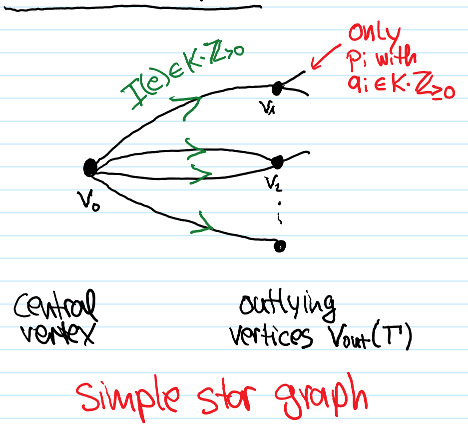
if we understand those
we can set up recursion.

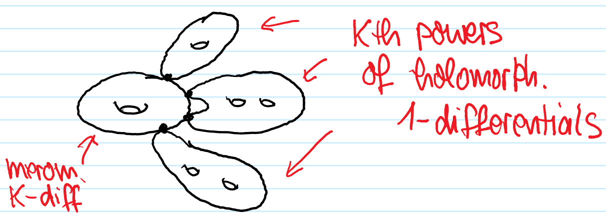
<u>Thm</u> (FP'15, S'16) $K \geq 1$

$Z \subset \widetilde{H}^K_g(A)$ component

<u>Generic Γ', I</u>

<u>Generic $(C, p_1, \dots, p_n)$</u>





$$\boxed{m_Z = \frac{\prod_{e \in E(\Gamma')} I(e)}{K^{\# V_{out}(\Gamma')}}}$$

Let $\xi_{\Gamma'} : \prod_{v \in V(\Gamma')} \overline{M}_{g(v), n(v)} \longrightarrow \overline{M}_{g,n}$ be the gluing map for Γ'.

$$\Rightarrow \bigcup_{\substack{Z \text{ w/ generic} \\ \Gamma', I}} Z = \xi_{\Gamma'}\left(\widetilde{H}^K_{g(v_0)}(A[v_0], -I[v_0] - K) \times \prod_{v \in V_{out}(\Gamma')} \widetilde{H}^1_{g(v)}\left(\frac{A[v]}{K}, \frac{I[v]-K}{K}\right)\right)$$

Conjecture A

Conjecture A (explicit version)

For $k \geq 1$ and $A \neq kA'$ for $A' \in \mathbb{Z}^n_{\geq 0}$, we have

$$\sum_{\Gamma \text{ star graph}} \sum_{I} \frac{\prod_{e \in E(\Gamma)} I(e)}{k^{|V_{\mathrm{out}}(\Gamma)|}} \frac{1}{|\mathrm{Aut}(\Gamma)|} C_{\Gamma,I} = 2^{-g} P_g^{g,k}(\widetilde{A})$$

where

$$C_{\Gamma,I} = (\xi_\Gamma)_* \Big[\left[\overline{\mathcal{H}}^k_{g(v_0)}(A[v_0], -I[v_0] - k)\right] \cdot \prod_{v \in V_{\mathrm{out}}(\Gamma)} \left[\overline{\mathcal{H}}^1_{g(v)}\left(\frac{A[v]}{k}, \frac{I[v] - k}{k}\right)\right] \Big]$$

Application: Recursion for $[\overline{\mathcal{H}}_g^k(A)]$

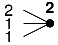
Theorem (JPPZ, S.)

Conjecture A effectively determines the classes $[\overline{\mathcal{H}}_g^k(A)]$ for

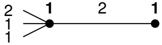
$$\begin{cases} k \geq 1 \text{ and } A \neq kA' \text{ with } A' \in \mathbb{Z}^n_{\geq 0} \text{ (codim } g) \\ k = 1 \text{ and } A \in \mathbb{Z}^n_{\geq 0} \text{ (codim } g-1) \end{cases}$$

Example: $[\overline{\mathcal{H}}_2^2(2,1,1)]$

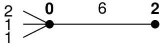
$$\left[\overline{\mathcal{H}}_2^2(2,1,1)\right]$$



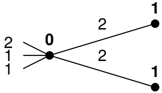
$$+\,(\xi_{\Gamma_2})_*\quad\left[[\overline{\mathcal{H}}_1^2(2,1,1,-4)]\cdot[\overline{M}_{1,1}]\right]$$



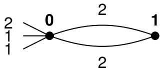
$$+\,3(\xi_{\Gamma_3})_*\quad\left[[\overline{M}_{0,4}]\cdot[\overline{\mathcal{H}}_2^1(2)]\right]$$



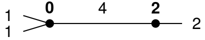
$$+\,\tfrac{1}{2}(\xi_{\Gamma_4})_*\quad\left[[\overline{M}_{0,5}]\cdot[\overline{M}_{1,1}]\cdot[\overline{M}_{1,1}]\right]$$



$$+\,(\xi_{\Gamma_5})_*\quad\left[[\overline{M}_{0,5}]\cdot[\overline{M}_{1,2}]\right]$$



$$+\,2(\xi_{\Gamma_6})_*\quad\left[[\overline{M}_{0,3}]\cdot[\overline{\mathcal{H}}_2^1(1,1)]\right]$$

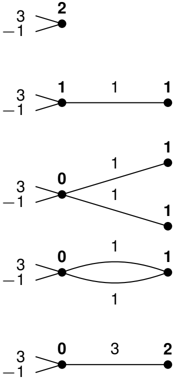


$$=\quad\frac{1}{4}P_2^{2,2}(4,3,3)$$

Example: $[\overline{\mathcal{H}}_2^1(2)]$

Take Conjecture A for $A^+ = (3, -1)$ on $\overline{M}_{2,2}$ and push forward under the forgetful map $\epsilon : \overline{M}_{2,2} \to \overline{M}_{2,1}$ of the second point

$$
\begin{aligned}
&\epsilon_*(\xi_{\Gamma_2})_* && \left[[\overline{\mathcal{H}}_1^1(3,-1,-2)] \cdot [\overline{M}_{1,1}]\right] \\
+&\tfrac{1}{2}\epsilon_*(\xi_{\Gamma_3})_* && \left[[\overline{M}_{0,4}] \cdot [\overline{M}_{1,1}] \cdot [\overline{M}_{1,1}]\right] \\
+&\tfrac{1}{2}\epsilon_*(\xi_{\Gamma_4})_* && \left[[\overline{M}_{0,4}] \cdot [\overline{M}_{1,2}]\right] \\
+&3 && \left[\overline{\mathcal{H}}_2^1(2)\right] \\
&= && \tfrac{1}{4}\epsilon_* P_2^{2,1}(4,0)
\end{aligned}
$$



Implementation in `admcycles`

The recursive algorithm to compute the classes $[\overline{\mathcal{H}}_g^k(A)]$ has been implemented in the SageMath package `admcycles` (with Vincent Delecroix, Jason van Zelm).

Applications

- [Sauvaget '20] Example computations of volumes of moduli spaces of flat surfaces,
- [Castorena-Gendron '20] Verified computation of $\pi_*[\overline{\mathcal{H}}_3^1(6,-2)] \in \mathrm{CH}^1(\overline{\mathcal{M}}_3)$
- [Costantini-Möller-Zachhuber] Ongoing project computing Euler characteristics of strata of differentials using intersection theory

Click here for an example computation.

§3 Chow group of $\mathfrak{Pic}_{g,0}$

$\rightsquigarrow$ use operational / bivariant / Chow cohom. approach (Fulton Chap. 17)

Let S be finite type scheme

$$S \xrightarrow{\varphi} \mathfrak{Pic}_{g,0} \quad \overset{\text{Def.}}{\Longleftrightarrow} \quad \begin{matrix} C & \mathcal{L} \\ \downarrow \uparrow p_i & \\ S & \end{matrix} \quad \text{family of curves} + \text{line bundle}$$

An operat. class $\alpha \in CH^c_{op}(\mathfrak{Pic}_{g,0})$ is data of

$$\Big(\alpha(\varphi) : CH_*(S) \longrightarrow CH_{*-c}(S) \Big)_{\varphi : S \to \mathfrak{Pic}_{g,0}}$$
$$\beta \longmapsto (\varphi^*\alpha) \cap \beta \qquad \uparrow \text{all such morph.}$$

Compatible with prop. pushforw, flat pullback, Gysin pullback

With some work:

$$\overline{e} \subset \mathfrak{Pic}_{g,0} \rightsquigarrow [\overline{e}] \in CH^g_{op}(\mathfrak{Pic}_{g,0})$$

$\uparrow$ closed, pure codim g

"Poincaré dual of fund class"

§4 The tautological ring of $\mathfrak{Pic}_{g,0}$

Idea
- Define $R^*(\mathfrak{Pic}_{g,0}) \subseteq CH^*_{op}(\mathfrak{Pic}_{g,0})$
- Express $[\overline{e}]$ as elem. in $R^*(\mathfrak{Pic}_{g,0})$

- $\mathcal{C} \leftarrow \mathcal{L} \leftarrow \exists$ universal line bundle!

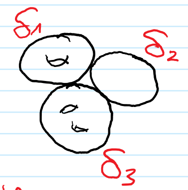
$\downarrow F$

$\mathfrak{Pic}_{g,0}$

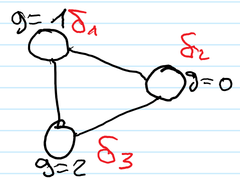
$$\rightsquigarrow \eta := F_*(c_1(\mathcal{L})^2) \in CH^1_{op}(\mathfrak{Pic}_{g,0})$$

- boundary strata of $\mathfrak{Pic}_{g,0}$ $\longleftrightarrow$ $\left.\begin{matrix}\text{prestable graphs } \Gamma \\ + \text{degree fct. } \delta : V(\Gamma) \to \mathbb{Z}\end{matrix}\right\} \Gamma_\delta$

$\sum \delta_i = 0$

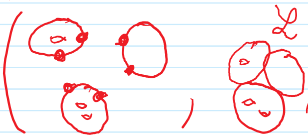


deg $\mathcal{L}$



- Given T'_δ have gluing morphism

$$j_{T'_\delta} : \mathrm{Pic}_{T'_\delta} \longrightarrow \mathrm{Pic}_{g,0}$$
$$\downarrow$$
$$\prod_{v \in V(T')} \mathrm{Pic}_{g(v),n(v),\delta(v)}$$

$\mathrm{Pic}_{T'_\delta} = \Big\{ \Big($  $\Big) \Big\}$

- Given T'_δ, $r \in \mathbb{Z}_{>0}$, an admissible weighting mod r is a map $w : \underbrace{H(T')}_{\text{half-edges of } T'} \longrightarrow \{0, \dots, r-1\}$ such that

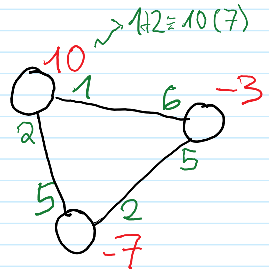
  → (h,h') form edge $\Rightarrow w(h)+w(h') \equiv 0 \pmod r$

  → $\forall v$ vertex : $\displaystyle\sum_{h \text{ at } v} w(h) \equiv \delta(v) \pmod r$

<u>Exa</u> $r=7$

T'_δ:

w

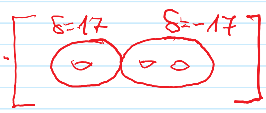


- <u>Def</u> For $r \in \mathbb{Z}_{>0}$, consider the mixed degree class in $CH^*_{op}(\mathrm{Pic}_{g,0})$ defined by

$$\underbrace{\exp\left(-\tfrac{1}{2}\eta\right)}_{=\left(1-\frac{1}{2}\eta+\frac{1}{2}\left(-\frac{1}{2}\eta\right)^2+\frac{1}{3!}\left(-\frac{1}{2}\eta\right)^3+\dots\right)} \cdot \sum_{T'_\delta} \sum_{w} \underbrace{\frac{r^{-h^1(T')}}{|\mathrm{Aut}(T'_\delta)|}}_{\substack{\text{combinator.}\\ \text{factor } \in \mathbb{Q}_{>0}}} \cdot \left(j_{T'_\delta}\right)_* \Bigg(\underbrace{\prod_{e=(h,h') \in E(T')} \frac{1-\exp\left(-\frac{w(h)w(h')}{2}(\psi_h+\psi_{h'})\right)}{\psi_h+\psi_{h'}}}_{\text{class on } \mathrm{Pic}_{T'_\delta}} \Bigg)$$

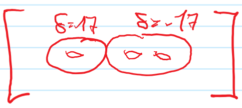
$$\frac{1-\exp(-a\cdot z)}{z} = a - \frac{a^2}{2}z + \frac{a^3}{3!}z^2 + \dots$$

Let $P_g^{c,r}$ be its codim. c part. For $r \gg 0$, this is polynomial in r.

<u>Exa</u> $P_{g=3}^{c=1,r} = \dots + \dfrac{17(r-17)}{2} \cdot \Big[$  $\Big] + \dots$

$(j_{T'_\delta})_*(1)$

Let $P_g^c = P_g^{c,r}\big|_{r=0} \in CH^c_{op}(\mathrm{Pic}_{g,0})$ be the value at $r=0$.

<u>Exa</u> $P_{g=3}^{c=1} = \dots + \dfrac{17(-17)}{2} \cdot \Big[$  $\Big] + \dots$

Thm (BHPSS '20)

We have $[\overline{e}] = P_g^g \in CH^g_{op}(\mathfrak{Pic}_{g,0})$.

§5 DR cycles with targets

[JPPZ] X nonsing. proj. variety, $\mathcal{L} \to X$ line bundle

Given $\beta \in H_2(X, \mathbb{Z})$:

$$\overline{\mathcal{M}}_{g,n,\beta}(X) = \{ (C, p_1, \ldots, p_n) \xrightarrow{f} X \}$$ stable maps of degree β

prestable

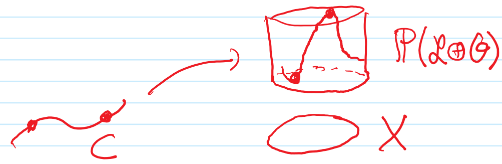
every comp. of C not contract. by f is stable.

Given $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ w/ $\sum a_i = \int_\beta c_1(\mathcal{L})$, the paper [JPPZ] defines a DR cycle $DR_{g,A,\beta}(X, \mathcal{L})$ on $\overline{\mathcal{M}}_{g,n,\beta}(X)$ compactifying the condition

$$f^* \mathcal{L} \cong \mathcal{O}_C\left(\sum a_i p_i\right)$$

idea use moduli space of maps to $\mathbb{P}(\mathcal{L} \oplus \mathcal{O}) \to X$

They show a Pixton-style formula $P^g_{g,A,\beta}(X, \mathcal{L})$ for $DR_{g,A,\beta}(X, \mathcal{L})$ ← use localization by $\mathbb{C}^*$-act. on $\mathbb{P}(\mathcal{L} \oplus \mathcal{O})$



What we can show:

$$\varphi : \overline{\mathcal{M}}_{g,n,\beta}(X) \longrightarrow \mathfrak{Pic}_{g,0}$$

$$\left((C, p_1, \ldots, p_n) \xrightarrow{f} X\right) \longmapsto \left(C, f^*\mathcal{L}\left(-\sum a_i p_i\right)\right)$$

$\overset{\text{Thm}}{\Rightarrow}$ $\varphi^*([\overline{e}]) \cap [\overline{\mathcal{M}}_{g,n,\beta}(X)]^{vir} = DR_{g,A,\beta}(X, \mathcal{L})$

$\varphi^*(P_g^g) \cap \quad \text{-- -- --} \quad = P^g_{g,A,\beta}(X, \mathcal{L})$)) [JPPZ]

Idea of Proof of main Theorem

For $X = \mathbb{P}^n$, $\beta = d \cdot [H]$ we can use the maps φ above as "charts" of $\mathfrak{Pic}_{g,0}$

- ↳ Known equal. from [JPPZ] $\Rightarrow$ $[\tilde{e}]$, P_g^g act in same way on $[\cdot]^{vir}$ via φ
- ↳ for $n, d \gg 0$, there is large open subs. of $\overline{M}_{g,n,\beta}(X)$ on which virt. fund. class = usual fund. class
- ↳ verify that knowing act. of $[\tilde{e}]$, P_g^g on these is enough to show equality in $CH^g_{op}(\mathfrak{Pic}_{g,0})$.

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