Summer term 2024 Dr. Andreas Mihatsch
Elliptic curves and their moduli spaces

Exam dates: August 5 (1st), and September 16 (2nd). Both at 9 am. Location will be announced in due time.

Homework problems (due June 28)

Problem 1 (Integral structure of $\mathrm{End}(E)$**)**

(a) Let ℓ be a prime. Consider the $\mathbb{Z}_\ell$-algebra

$$R = \left\{ \begin{pmatrix} a & b \\ \ell c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{Z}_\ell \right\}.$$

Prove that $R \not\cong M_2(\mathbb{Z}_\ell)$.

(b) Let $L = \mathrm{Hom}(E_1, E_2)$ be the Hom-space between two elliptic curves over a field k. Let $\ell \neq \mathrm{char}(k)$ be a prime. Show that $\phi \in L$ is divisible by ℓ in L if and only if it is divisible by ℓ in $\mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell(E_1), T_\ell(E_2))$.

(c) Assume that $\mathrm{char}(k) = p$ and that E/k is such that $\mathrm{End}^0(E)$ is a quaternion algebra. Prove that

$$\mathbb{Z}_\ell \otimes_{\mathbb{Z}} \mathrm{End}(E) \cong M_2(\mathbb{Z}_\ell).$$

In particular, subrings as in (a) cannot occur as ℓ-adic completions of $\mathrm{End}(E)$.

Problem 2 (On elliptic curves with complex multiplication)

Let K be an imaginary-quadratic field. For $i = 1, 2$, let E_i be a complex elliptic curve with $\mathrm{End}^0(E_i) \cong K$. Show that $\mathrm{Hom}(E_1, E_2) \neq 0$.

Further Problems

Problem 3 (A p**-adic argument)**

Let k be a field of characteristic $p > 0$ and let E be an elliptic curve over k.

(a) Show that there are natural projection maps $\mathrm{End}(E[p^{n+1}]) \to \mathrm{End}(E[p^n])$.

(b) Show that there is a natural map

$$\mathrm{End}(E) \longrightarrow \varprojlim \mathrm{End}(E[p^n]).$$

Prove that this map is injective.

Hint: Follow the strategy of the ℓ*-adic case from the lecture.*