Exercise Sheet 11

Discussed on 07.07.2021

Problem 1. Let k be an algebraically closed field and C a proper smooth connected curve over k. We will assume that the relative Picard functor $\mathrm{Pic}^0_{C/k}$ is representable by a k-scheme which is locally of finite type. The goal is to show that $\mathrm{Pic}^0_{C/k}$ is an abelian variety of dimension $g := g(C)$.

(a) Let X be a scheme. Show that there is a canonical bijection

$$\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times).$$

(b) Show that the tangent space of $\mathrm{Pic}^0_{C/k}$ at 0 equals $H^1(C, \mathcal{O}_C)$ and thus has dimension g.

(c) Show that $\mathrm{Pic}^0_{C/k}$ is smooth over k.

(d) Fix a point $P \in C(k)$. Show that there is a map $\varphi\colon C^g \to \mathrm{Pic}^0_{C/k}$ which on k-points is given by $(P_1, \dots, P_g) \mapsto \mathcal{O}_C([P_1] + \dots + [P_g] - g[P])$.

(e) Prove that the map φ is surjective. Deduce that $\mathrm{Pic}^0_{C/k}$ is proper and connected.