Consistency strength results about mutual stationarity

Forcing $\mathrm{MS}(\aleph_3, \aleph_5, \aleph_7, \ldots; \omega_1)$

Peter Koepke, University of Bonn

Logic in Hungary, Budapest, August 2005

A model theoretic characterization of stationarity

For $\kappa \geqslant \aleph_1$ regular and $S \subseteq \kappa$ the following are equivalent:

- S is stationary in κ
- every first-order structure $\mathfrak{A} \supseteq \kappa$ with countable language has an elementary substructure $X \prec \mathfrak{A}$ such that $X \cap \kappa \in S$
- every first-order structure $\mathfrak{A} \supseteq \kappa$ with countable language has an elementary substructure $X \prec \mathfrak{A}$ such that $\sup(X \cap \kappa) \in S$

Mutual stationarity

(S_i) is mutually stationary in (κ_i) if every first-order structure $\mathfrak{A} \supseteq \bigcup_i \kappa_i$ with countable language has an elementary substructure $X \prec \mathfrak{A}$ such that $\forall i \sup(X \cap \kappa_i) \in S_i$.

Obviously: if (S_i) is mutually stationary in (κ_i) then $\forall i\ S_i$ is stationary in κ_i.

The mutual stationarity problem (Foreman, Magidor):

(When) does the converse hold?

Mutual Ramseyness

Consider regular cardinals

$$\kappa_0 < \kappa_1 < \ldots < \kappa_n < \ldots, n < \omega, \kappa = \sup \kappa_n$$

(κ_n) is mutually Ramsey (coherently Ramsey) if for all $F\colon [\kappa]^{<\omega} \to 2$ there are sets $A_n \subseteq \kappa_n$, $\mathrm{card}(A_n) = \kappa_n$ such that (A_n) is homogeneous for F:

for all $x, y \subseteq \bigcup A_n$, x, y finite, $\forall n < \omega$ $\mathrm{card}(x \cap A_n) = \mathrm{card}(y \cap A_n)$ holds

$$F(x) = F(y).$$

The sequence (A_n) is mutually indiscernible for a structure coded by F (all structures are assumed to have built-in Skolem functions).

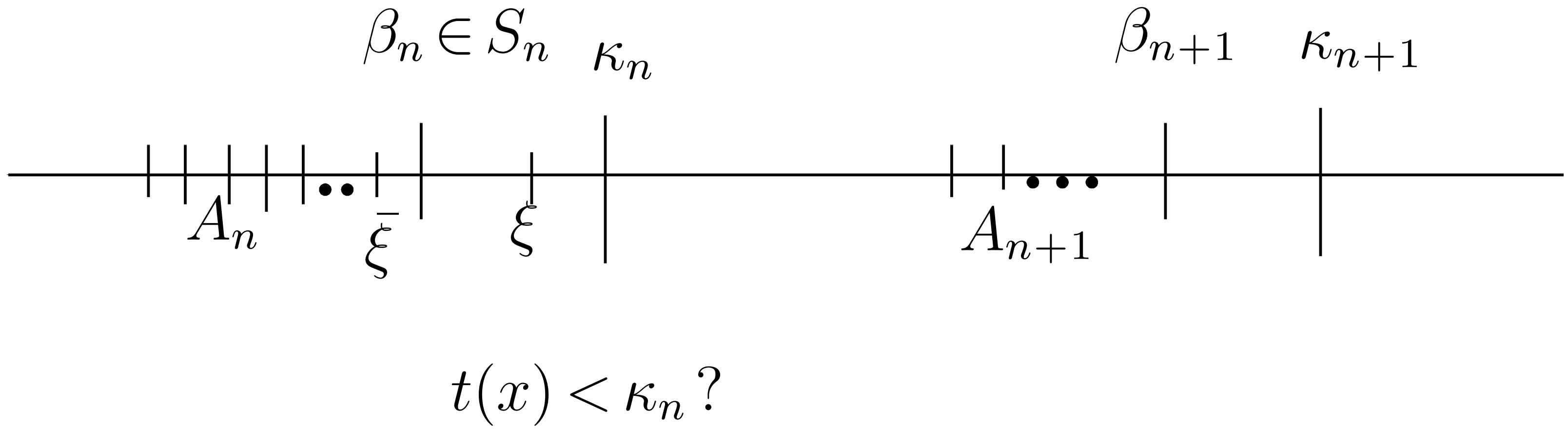
Mutual stationarity from mutual indiscernibles

Theorem. Let (κ_n) be mutually Ramsey. Then the mutual stationarity property $\mathrm{MS}(\kappa_0, \kappa_1, \ldots)$ holds: if $\forall n < \omega\, S_n$ is stationary in κ_n then (S_n) is mutually stationary in (κ_n).

Proof. Let (A_n) be mutually indiscernible for a given structure $\mathfrak{A} \supseteq \kappa$. Let $\beta_n \in S_n$, $\sup\,(A_n \cap \beta_n) = \beta_n$. Let X be the elementary substructure of $\mathfrak{A}$ generated by $\bigcup_{n<\omega} (A_n \cap \beta_n)$. Then

$$\beta_n \leqslant \sup\,(X \cap \kappa_n) \leqslant \beta_n\,.$$

Let $t^{\mathfrak{A}}(x) = t^{\mathfrak{A}}(x \cap \kappa_n, x \setminus \kappa_n) < \kappa_n$. Let $t^{\mathfrak{A}}(x) = t^{\mathfrak{A}}(x \cap \kappa_n, x \setminus \kappa_n) < \xi$, $\xi \in A_n \cap \kappa_n$. By indiscernibility, $t^{\mathfrak{A}}(x) = t^{\mathfrak{A}}(x \cap \kappa_n, x \setminus \kappa_n) < \bar{\xi} < \beta_n$ for some $\bar{\xi} \in A_n \cap \beta_n$.


$\beta_n \in S_n$
κ_n
β_{n+1}
κ_{n+1}
A_n
$\bar{\xi}$
ξ
A_{n+1}
$t(x) < \kappa_n$?

Consistency strengths

κ measurable

$\Downarrow$ *Prikry forcing*

endsegment of a Prikry sequence (κ_n) is mutually Ramsey

$\Downarrow$

$\mathrm{MS}(\kappa_0, \kappa_1, \ldots)$ (Cummings, Foreman, Magidor)

$\Downarrow$

κ is a singular Jónsson cardinal

$\Downarrow$ *inner models*

κ is measurable in an inner model (Mitchell)

Accessible κ_i's

$\mathrm{MS}(\aleph_1, \aleph_2, \ldots) \to \aleph_\omega$ is Jónsson $\to$???

Restricting cofinalities

The mutual stationarity property in cofinality γ (Foreman, Magidor):

$\mathrm{MS}(\kappa_0, \kappa_1, \ldots; \gamma)$: if $\forall n < \omega\, S_n \subseteq \mathrm{cof}_\gamma$ is stationary in κ_n then (S_n) is mutually stationary in (κ_n).

Foreman, Magidor:

$\mathrm{ZFC} \vdash \mathrm{MS}(\kappa_0, \kappa_1, \ldots; \omega)$

K., Welch:

$\mathrm{MS}(\kappa_0,\ \kappa_1,\ \ldots;\ \omega_1) \rightarrow$ there is an inner model with one measurable cardinal

K., Welch:

$\mathrm{MS}(\aleph_2,\ \aleph_3,\ \ldots;\ \omega_1) \rightarrow$ there is an inner model with many measurable cardinals

No upper consistency bound for $\mathrm{ZFC} + \mathrm{MS}(\aleph_2, \aleph_3, \ldots; \omega_1)$ is known.

Main Theorem (K.)

Let (κ_n) be mutually Ramsey with supremum κ. Then there is a generic extension $V[G]$ such that

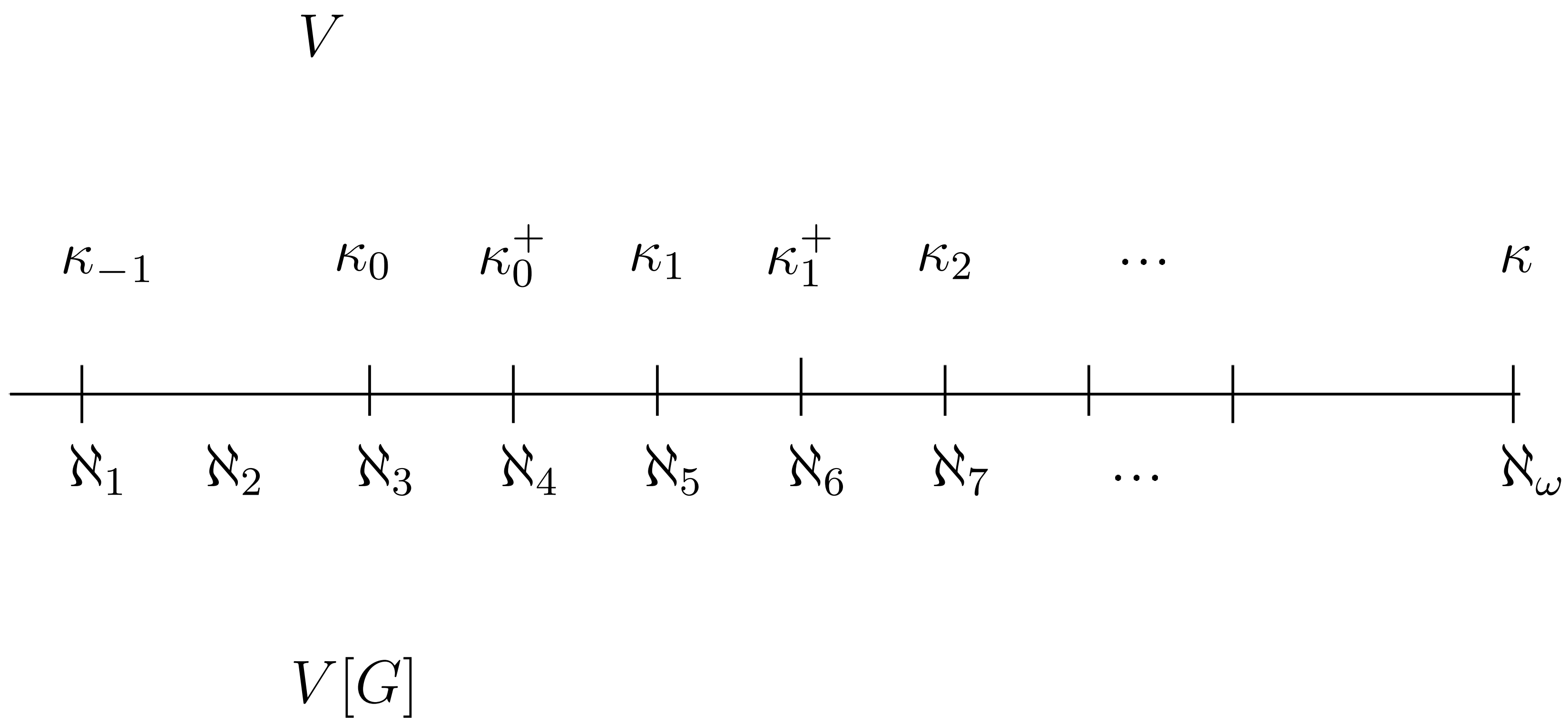
$$V[G] \models \mathrm{MS}(\aleph_3, \aleph_5, \aleph_7, \ldots; \omega_1).$$

Elements of the **Proof**. Let

$$P = \prod_{n<\omega} \mathrm{Col}(\kappa_{n-1}^{+}, <\kappa_n), \text{ where } \kappa_{-1} = \aleph_1 .$$

Every $p \in P$ is of the form $p = (p_n | n < \omega)$.

Let G be P-generic over V.


V
κ_{-1} κ_0 κ_0^+ κ_1 κ_1^+ κ_2 $\cdots$ κ
$\aleph_1$ $\aleph_2$ $\aleph_3$ $\aleph_4$ $\aleph_5$ $\aleph_6$ $\aleph_7$ $\cdots$ $\aleph_\omega$
$V[G]$

Let $(\kappa, F) \in V[G]$, $F\colon [\kappa]^{<\omega} \to \kappa$.
Let $F = \dot{F}^G$.
Let $p \in P$, $p \Vdash \dot{F}\colon [\kappa]^{<\omega} \to \kappa$.

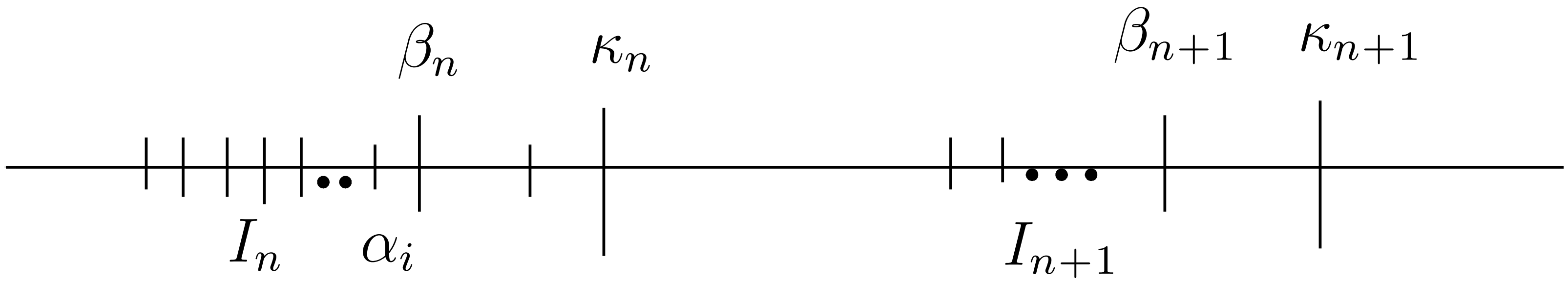
Fixing suprema
Let (A_n) be "good" mutual indiscernibles for $(V_\theta, \in, \ldots, \dot{F}, p, (\dot{S}_n))$.
Let $I_n \subseteq A_n$, $\mathrm{otp}(I_n) = \omega$, $\sup(I_n \cap \beta_n) = \beta_n$.

Let $[\bigcup I_n]^{<\omega} = \{x_i | i < \omega\}$.
Construct a "generic sequence"

$$p \geqslant p(x_0) \geqslant p(x_1) \geqslant \ldots$$

deciding the terms $\dot{F}(x_i)$:

$$p(x_i) \Vdash \dot{F}(x_i) = \alpha_i \in \mathrm{Ord}$$



$$\alpha_i = t(x_0, x_1, \ldots, x_i) < \kappa_n \longrightarrow \alpha_i < \beta_n$$

Let $q = \bigcup p(x_i)$ be the coordinatewise union of $(p(x_i))$:

$$q_n = \bigcup_{i<\omega} p_n(x_i)\,.$$

Let $X = \{\alpha_i | i < \omega\}$. Then

$$q \Vdash \check{X} \prec (\kappa, \dot{F}) \wedge \sup(\check{X} \cap \kappa_n) = \beta_n\,.$$

Meeting stationary sets

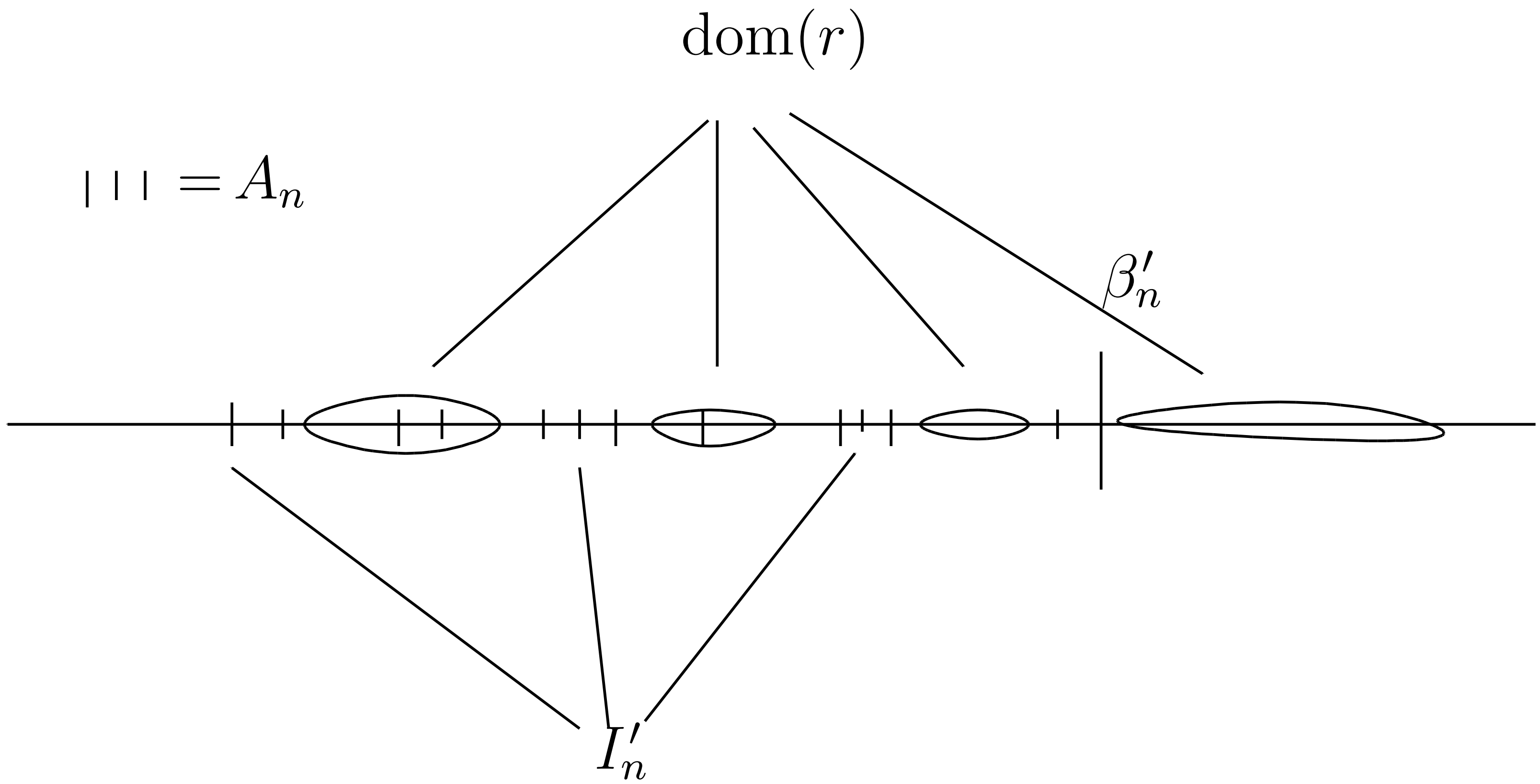
Let $V[G] \models S_n = \dot{S}_n^G \subseteq \mathrm{cof}_\omega$ is stationary in κ_n. Assume

$$p \Vdash \dot{S}_n \subseteq \mathrm{cof}_\omega \text{ is stationary in } \kappa_n .$$

Let $\beta_n' \in S_n$ be a high-level limit of A_n. Let $r \leqslant q$ such that

$$r \Vdash \beta_n' \in \dot{S}_n .$$

Choose $I_n' \subseteq A_n$, $\mathrm{otp}(I_n') = \omega$, $\sup(I_n' \cap \beta_n') = \beta_n'$, so that I_n' "lies apart" from the condition r:



$\mathrm{dom}(r)$
$| \; | \; | = A_n$
β'_n
I'_n

The system (I'_n) is order-isomorphic to (I_n). By this isomorphism let

$$x'_i \cong x_i\,,\, p(x'_i) \cong p(x_i)\,,\; q' \cong q\,,\; \alpha'_i \cong \alpha_i\,,\; X' \cong X$$

By indiscernibility,

$$q' \Vdash \breve{X}' \prec (\kappa, \dot{F}) \wedge \sup(\breve{X}' \cap \kappa_n) = \beta'_n\,.$$

By the choice of (I'_n), q' is compatible with r. Hence

$$q' \cup r \Vdash \breve{X}' \prec (\kappa, \dot{F}) \wedge \sup(\breve{X}' \cap \kappa_n) \in \dot{S}_n\,.$$

This is a forcing construction for the Foreman-Magidor ZFC-result:

$$V[G] \vDash \mathrm{MS}(\aleph_3, \aleph_5, \aleph_7, \ldots; \omega).$$

From cof_ω *to* cof_{ω_1}

Fixing a substructure of size ω $\equiv$ Rasiowa-Sikorski construction of a generic filter for countably many dense sets.

Fixing a substructure of size ω_1 $\overset{?!}{\equiv}$ getting a generic filter for ω_1 dense sets via Martin's axiom MA_{ω_1} like in Silver's forcing construction of Chang's conjecture.

Assume $V \vDash \mathrm{MA}_{\omega_1}$ (by small forcing).

Let $I_n \subseteq A_n$, $\mathrm{otp}(I_n) = \omega_1$, $\sup(I_n \cap \beta_n) = \beta_n$.

Let $[\bigcup I_n]^{<\omega} = \{x_i | i < \omega_1\}$.

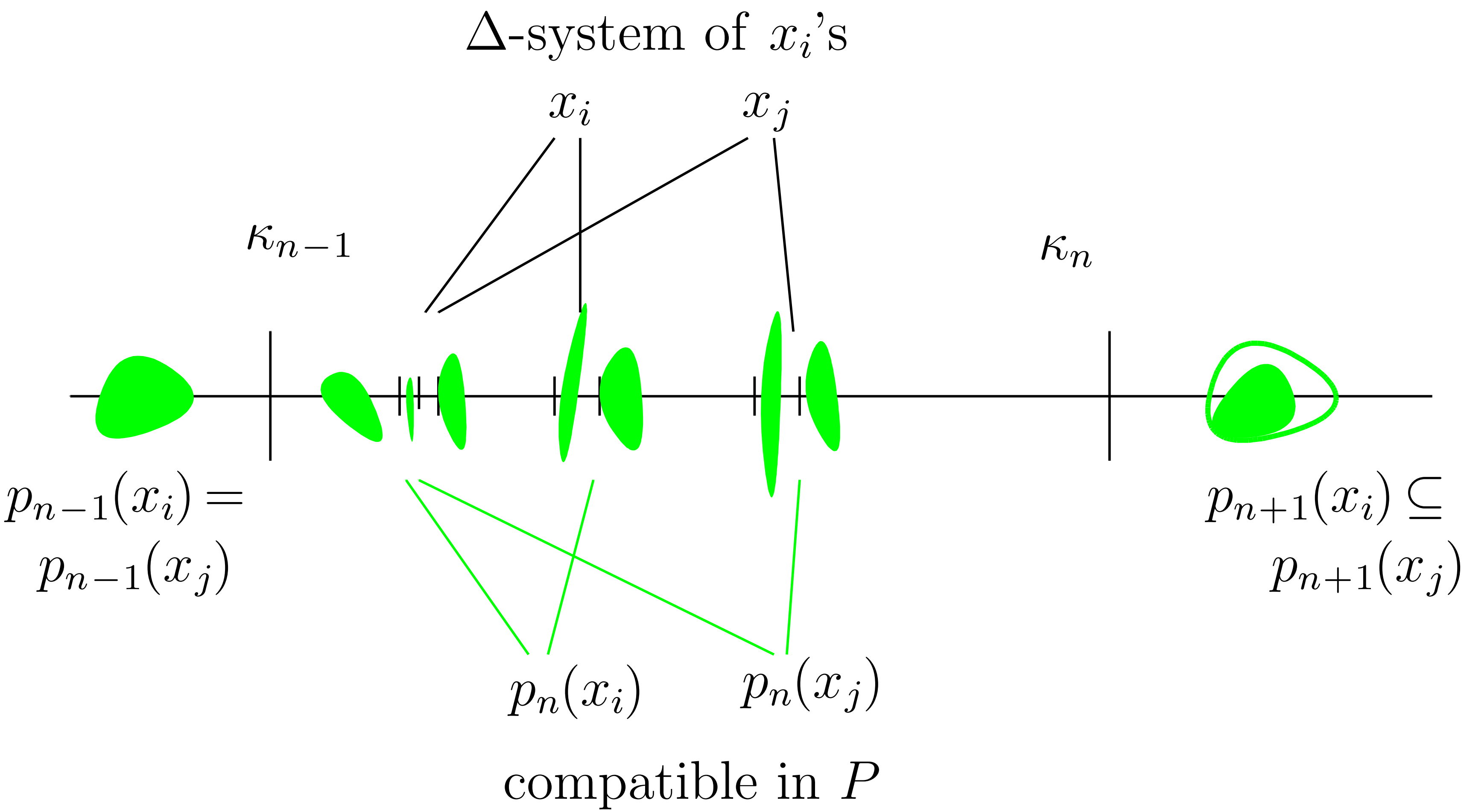
Dense sets

$$D_i = \{s \mid \exists \alpha \, s \Vdash \dot{F}(x_i) = \check{\alpha}\}.$$

But: P does not have the countable chain condition (ccc).

Constructing a suitable ccc $Q \subseteq P$

Silver: Let $Z \prec (V_\theta, \in, \ldots, \dot{F}, p, (\dot{S}_n))$ be generated by $\bigcup I_n$, and let $Q = Z \cap P$.


Δ-system of x_i's
x_i
x_j
κ_{n-1}
κ_n
$p_{n-1}(x_i) =$
$p_{n-1}(x_j)$
$p_{n+1}(x_i) \subseteq$
$p_{n+1}(x_j)$
$p_n(x_i)$
$p_n(x_j)$
compatible in P

For the ccc-argument, consider some Δ-system of x_i's in the interval (κ_{n-1}, κ_n):

- for $m < n$, $p_m(x_i) = p_m(x_j)$ by indiscernibility;
- for $m = n$, $p_n(x_i)$ is compatible with $p_n(x_j)$ by a standard ccc-argument;
- for $m > n$, $p_m(x_i) \subseteq p_m(x_j)$ by some "growth condition".

Construct the suborder $Q = \{p(x_i) | i < \omega_1\}$ such that $D_i \cap Q$ is dense in Q.

By MA_{ω_1} let H be Q-generic over $\{D_i \cap Q | i < \omega_1\}$.

Let $q = \bigcup H$ (coordinatewise).

Let $X = \{\alpha \,|\, \exists i < \omega_1 \; q \Vdash \dot{F}(x_i) = \check{\alpha}\}$. Then

$$q \Vdash \sup (\check{X} \cap \kappa_n) = \beta_n\,.$$

As before, we can also choose the I_n sufficiently apart from a condition r which fixes $\beta_n \in S_n$. Then

$$q \cup r \Vdash \sup (\check{X} \cap \kappa_n) \in \dot{S}_n\,.$$

Hence

$$V[G] \vDash \mathrm{MS}(\aleph_3, \aleph_5, \aleph_7, \ldots; \omega_1).$$

Variants

– $\mathrm{MS}(\aleph_{n(0)}, \aleph_{n(1)}, \aleph_{n(2)}, \ldots; \omega_1)$, where

$$\exists i_0 < \omega \forall i \geqslant i_0 \;\; n(i+1) \geqslant n(i) + 2.$$

– $\mathrm{MS}(\aleph_{n(0)}, \aleph_{n(1)}, \aleph_{n(2)}, \ldots; \omega/\omega_1)$, where

$$\exists i_0 < \omega \forall i \geqslant i_0 \;\; n(i+1) \geqslant n(i) + 2.$$

The forcing method does not go above cofinality ω_1:

$\mathrm{MS}(\aleph_3, \aleph_5, \aleph_7, \ldots; \omega_2) \rightarrow$ there is an inner model with many measurable cardinals.

Conjecture

The consistency strength of

$$\mathrm{MS}(\aleph_1, \aleph_2, \aleph_3, \aleph_4, \ldots; \omega, \omega_1, \omega, \omega_1, \ldots)$$

is the existence of 1 measurable cardinal.