FORCING A MUTUAL STATIONARITY PROPERTY IN COFINALITY ω_1

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ABSTRACT. We show that the consistency strength, relative to the system ZFC, of the mutual stationarity property $MS(\aleph_3, \aleph_5, \aleph_7, \ldots; \omega_1)$ is equal to the existence of one measurable cardinal. We also discuss mutual stationarity for some other configurations of small cardinal parameters.

1. INTRODUCTION

The concept of *mutual stationarity*, introduced by M. Foreman and M. Magidor [1], allows us to transfer some combinatorial aspects of stationary sets to singular cardinals.

Definition 1.1. Let $(\kappa_n)_{n<\omega}$ be a strictly increasing sequence of regular cardinals $\geqslant \aleph_2$ with $\kappa_\omega = \sup_{n<\omega} \kappa_n$. A sequence $(S_n)_{n<\omega}$ is called *mutually stationary* in $(\kappa_n)_{n<\omega}$ if every first-order structure $\mathfrak{A}$ of countable type with $\kappa_\omega \subseteq \mathfrak{A}$ has an elementary substructure $B \prec \mathfrak{A}$ such that $\forall n < \omega \sup B \cap \kappa_n \in S_n$.

If $(S_n)_{n<\omega}$ is mutually stationary in $(\kappa_n)_{n<\omega}$, then, by a standard model-theoretic characterisation of stationarity, each $S_n \cap \kappa_n$ is stationary in κ_n. For the study of the converse one formulates a *mutual stationarity property* MS. This property is dependent on the cofinalities of the ordinals in the stationary sets S_n; we denote the class $\{\xi \in \mathrm{Ord} \mid \mathrm{cof}(\xi) = \lambda\}$ by Cof_λ.

Definition 1.2. Let $(\kappa_n)_{n<\omega}$ be a strictly increasing sequence of regular cardinals and let $(\lambda_n)_{n<\omega}$ be a sequence of regular cardinals such that $\forall n < \omega \; \lambda_n < \kappa_n$. The *mutual stationarity property* $MS((\kappa_n)_{n<\omega}; (\lambda_n)_{n<\omega})$ is the statement: if $(S_n)_{n<\omega}$ is a sequence of sets $S_n \subseteq \mathrm{Cof}_{\lambda_n}$ each of which is stationary in κ_n, then $(S_n)_{n<\omega}$ is mutually stationary in $(\kappa_n)_{n<\omega}$. If $(\lambda_n)_{n<\omega}$ is the constant sequence $\lambda_n \equiv \lambda$ we also write $MS((\kappa_n)_{n<\omega}; \lambda)$.

In cofinality ω there is a remarkable ZFC result by M. Foreman and M. Magidor [1], whereas in higher cofinalities, and for accessible cardinals κ_n, the mutual stationarity property gains large cardinal strength.

Theorem 1.3. *For $(\kappa_n)_{n<\omega}$ a strictly increasing sequence of uncountable regular cardinals, $MS((\kappa_n)_{n<\omega}; \omega)$ holds.*

Theorem 1.4 (J. Cummings, M. Foreman and M. Magidor [2]; P. Koepke and P. Welch [5]). *The theories* ZFC $+ \exists(\kappa_n)_{n<\omega}\mathrm{MS}((\kappa_n)_{n<\omega};\omega_1)$ *and* ZFC $+ \exists\kappa\ \kappa$ *measurable are equiconsistent.*

Theorem 1.5 (Koepke and Welch [5])**.**

a) *The property* $\mathrm{MS}(\aleph_2, \aleph_3, \aleph_4, \ldots; \omega_1)$ *implies that there is an inner model with an inaccessible limit of measurable cardinals.* b) *The property* $\mathrm{MS}(\aleph_3, \aleph_5, \aleph_7, \ldots; \omega_2)$ *implies that there is an inner model with an inaccessible limit of measurable cardinals.*

The main result of this paper says that there are, nevertheless, some mutual stationarity properties in cofinality ω_1 for particular sequences of $\aleph_n$'s which are equiconsistent with *one* measurable cardinal:

Theorem 1.6. *Let κ be a measurable cardinal. Then there is a forcing extension of V in which* $\mathrm{MS}(\aleph_3, \aleph_5, \aleph_7, \ldots; \omega_1)$ *holds.*

In the construction we adapt an argument of Silver's for forcing Chang's conjecture with the help of Martin's axiom $\mathrm{MA}_{\aleph_1}$ (see paragraph 19 of [3]). Other mutual stationarities hold in the extension. Previous authors have forced the following from much stronger large cardinals; see e.g. [6], [7], or [2].

Theorem 1.7. *In the above forcing extension* $(\mathrm{Cof}_\omega, \mathrm{Cof}_{\omega_1}, \mathrm{Cof}_\omega, \mathrm{Cof}_{\omega_1}, \ldots)$ *is mutually stationary in* $(\aleph_2, \aleph_3, \aleph_4, \aleph_5, \ldots)$.

2. The forcing construction

In our forcing extension, we shall prove a combinatorial principle which is equivalent to the model-theoretic characterisation of our mutual stationarity property. A standard coding of a structure $\mathfrak{A} \supseteq \aleph_\omega$ by a (Skolem) function $F : [\aleph_\omega]^{<\omega} \to \aleph_\omega$ yields:

Proposition 2.1. $\mathrm{MS}(\aleph_3, \aleph_5, \aleph_7, \ldots; \omega_1)$ *is equivalent to the property: for every function $F : [\aleph_\omega]^{<\omega} \to \aleph_\omega$ and every sequence $(S_n)_{n<\omega}$, S_n stationary in $\aleph_{2n+3}$, $S_n \subseteq \mathrm{cof}_{\omega_1}$ there exists a set $X \subseteq \aleph_\omega$ such that*

$$\forall n < \omega\ \sup((F''[X]^{<\omega}) \cap \aleph_{2n+3}) \leqslant \sup(X \cap \aleph_{2n+3}) \in S_n.$$

The following partition property will be crucial in the subsequent proof:

Definition 2.2. Let $\kappa_0 < \kappa_1 < \ldots$ be a sequence of infinite cardinals with supremum κ.

a) For $x \in [\kappa]^{<\omega}$ let $\mathrm{type}(x) = (\mathrm{card}(x \cap \kappa_n)|n < \omega) \in {}^\omega\omega$; $\mathrm{type}(x)$ is called the *type* of x. We say that $t \in {}^\omega\omega$ is a *type* if t is of the form $\mathrm{type}(x)$ for some $x \in [\kappa]^{<\omega}$.
b) If t is a type, let $[\kappa]^t = \{x \in [\kappa]^{<\omega}|\, \mathrm{type}(x) = t\}$.
c) For $a \subseteq \omega$ define a *projection function* $\upharpoonright a : [\kappa]^{<\omega} \to [\kappa]^{<\omega}$ by

$$x \upharpoonright a = \{\xi \in x|\, \mathrm{otp}(\xi \cap x) \in a\}.$$

d) Let $F : [\kappa]^{<\omega} \to \kappa$. A sequence $(I_n)_{n<\omega}$ is *mutually homogeneous* for F if for all $x, y \in [\bigcup_{n<\omega} I_n]^{<\omega}$ with $\mathrm{type}(x) = \mathrm{type}(y)$:

$$\text{if } x \cap (F(x) + 1) = y \cap (F(x) + 1), \text{ then } F(x) = F(y).$$

e) The sequence $(\kappa_n)_{n<\omega}$ is *mutually Ramsey* if for every function $F : [\kappa]^{<\omega} \to \kappa$ there exists a mutually homogeneous sequence $(I_n)_{n<\omega}$ such that $\operatorname{card}(X_n) = \kappa_n$ for all $n < \omega$.

Obviously, every element of a mutually Ramsey sequence $(\kappa_n)_{n<\omega}$ is a Ramsey cardinal. The consistency strength of the existence of a mutually Ramsey sequence $(\kappa_n)_{n<\omega}$ is that of one measurable cardinal. Theorem 3.2 of [4] shows:

Proposition 2.3. *Let κ be a measurable cardinal. Let $(\kappa_n)_{n<\omega}$ be a Prikry sequence adjoined by Prikry forcing with a measure on κ. Then there is an endsegment of $(\kappa_n)_{n<\omega}$ which is mutually Ramsey in the Prikry generic extension $V[(\kappa_n)_{n<\omega}]$.*

We can now turn to the

Proof of Theorem 1.6. By the previous proposition we may generically extend the ground model to a model in which there is a sequence $(\kappa_n)_{n<\omega}$ of mutually Ramsey cardinals with supremum κ. Using "small forcing" we may also assume that Martin's Axiom $\mathrm{MA}_{\aleph_1}$ holds in the ground model V. We shall turn $\kappa_0 < \kappa_1 < \kappa_2 \ldots < \kappa$ into $\aleph_3, \aleph_5, \aleph_7, \ldots, \aleph_\omega$ respectively by an appropriate product of Levy collapses. Set $\kappa_{-1} = \aleph_1$. Let

$$P = \prod_{n<\omega} \operatorname{Col}(\kappa_{n-1}^+, < \kappa_n),$$

where $\operatorname{Col}(\kappa_{n-1}^+, < \kappa_n)$ is the Levy collapse for collapsing κ_n to κ_{n-1}^{++}; every condition $r \in \operatorname{Col}(\kappa_{n-1}^+, < \kappa_n)$ satisfies $\operatorname{dom}(r) \subseteq \kappa_{n-1}^+ \times \kappa_n$. For $p \in P$ and $n < \omega$ write p_n instead of $p(n)$. P is partially ordered by coordinatewise reverse inclusion:

$$p \leqslant q \text{ iff } \forall n < \omega \ p_n \supseteq q_n.$$

Let G be P-generic over V. $V[G]$ is the intended generic extension, which we describe by a sequence of claims. The preservation of certain cardinals is proved by a product analysis of P. Let $m < \omega$ and define

$$P_{<m} = \prod_{n<m} \operatorname{Col}(\kappa_{n-1}^+, < \kappa_n) \text{ and } P_{\geqslant m} = \prod_{m \leqslant n<\omega} \operatorname{Col}(\kappa_{n-1}^+, < \kappa_n),$$

both ordered by coordinatewise reverse inclusion $\leqslant$. If $p \in P$, then $p \upharpoonright m \in P_{<m}$ and $p \upharpoonright (\omega \setminus m) \in P_{\geqslant m}$. Obviously, $P_{\geqslant 0} = P$.
  (1) $(P_{\geqslant m}, \leqslant)$ is $< \kappa_{m-1}^+$-complete.
  (2) $(P_{<m}, \leqslant)$ satisfies the κ_{m-1}-chain condition.   □

Proof. This is obvious in case $m = 0$. Let $m > 0$ and let $A \subseteq P_{<m}$ with $\operatorname{card}(A) = \kappa_{m-1}$. Every $p \in P_{<m}$ may be split into $p \upharpoonright (m-1) \in P_{<m-1}$ and $p(m-1) \in \operatorname{Col}(\kappa_{m-2}^+, < \kappa_{m-1})$. Since $\operatorname{card}(P_{<m-1}) < \kappa_{m-1}$ we may assume that $p \upharpoonright (m-1)$ is the same for all $p \in A$. Since $\operatorname{Col}(\kappa_{m-2}^+, < \kappa_{m-1})$ satisfies the κ_{m-1}-chain condition there must be $p, q \in A$, $p \neq q$ such that $p(m-1)$ and $q(m-1)$ are compatible in $\operatorname{Col}(\kappa_{m-2}^+, < \kappa_{m-1})$. Since $p \upharpoonright (m-1) = q \upharpoonright (m-1)$, p and q are compatible in $(P_{<m}, \leqslant)$. This proves (2).   □

(3) Let $p \in P$, $p \Vdash \dot{\alpha} \in \operatorname{Ord}$. Then there is some $w \subseteq \operatorname{Ord}$, $\operatorname{card}(w) < \kappa_{m-1}$ and some $q \leqslant p$ such that

$$q \Vdash \dot{\alpha} \in w \text{ and } q \upharpoonright m = p \upharpoonright m.$$

Proof. Define a sequence $(p^i | i < \theta)$ recursively such that

$$\theta \leqslant \kappa_{m-1} \text{ and } \forall i < j < \theta \ (p_i \leqslant p \wedge p_j \upharpoonright (\omega \setminus m) \leqslant p_i \upharpoonright (\omega \setminus m)).$$

Let $k < \kappa_{m-1}$ and let $(p^i | i < k)$ be defined satisfying the recursive condition. Define a condition $q^k \in P$ by

$$q_n^k = \begin{cases} p_n, \text{ if } n < m; \\ \bigcup_{i<k} p_n^i, \text{ if } n \geqslant m. \end{cases}$$

q^k is a condition by the closure properties of $P_{\geqslant m}$. If there is $p' \leqslant q^k$ and some $\gamma \in \mathrm{Ord}$ such that $p' \Vdash \dot{\alpha} = \check{\gamma}$ and $\forall i < k \ p' \upharpoonright m$ is incompatible with $p^i \upharpoonright m$, then let p^k be such a p' and continue. If not set $\theta = k$ and stop the recursion.

By construction, $(p^i \upharpoonright m \mid i < \theta)$ is an antichain in $P_{<m}$. Since $P_{<m}$ satisfies the κ_{m-1}-chain condition, $\theta < \kappa_{m-1}$. Define a condition $q \in P$ by

$$q_n = \begin{cases} p_n, \text{ if } n < m; \\ \bigcup_{i<\theta} p_n^i, \text{ if } n \geqslant m. \end{cases}$$

Let $w = \{\gamma \in \mathrm{Ord} \mid \exists i < \theta \ p^i \Vdash \dot{\alpha} = \check{\gamma}\}$. Then $\mathrm{card}(w) \leqslant \theta < \kappa_{m-1}$. We show that $q \Vdash \dot{\alpha} \in w$. Note that $q = q^\theta$ in the above recursion. Let $r \leqslant q$. Let $s \leqslant r$ and $\delta \in \mathrm{Ord}$ such that $s \Vdash \dot{\alpha} = \check{\delta}$. Since p^θ is not defined there is $i < \theta$ such that $s \upharpoonright m$ and $p^i \upharpoonright m$ are compatible. Then s and p^i are compatible. Let $\gamma \in w$ such that $p^i \Vdash \dot{\alpha} = \check{\gamma}$. By the compatibility of s and p^i we have $\delta = \gamma \in w$ and so $s \Vdash \dot{\alpha} \in \check{w}$. Thus $\{s | s \Vdash \dot{\alpha} \in \check{w}\}$ is dense below q and $q \Vdash \dot{\alpha} \in \check{w}$. This proves (3). $\qquad \square$

(4) Let $m < \omega$. Then κ_{m-1} and κ_{m-1}^+ are cardinals in $V[G]$.

Proof. Assume that $p \in P$ and $p \Vdash (\dot{h} : \check{\theta} \to \kappa_{m-1}$ is surjective) for some $\theta < \kappa_{m-1}$. By the previous claim we can recursively define a $\leqslant$-decreasing sequence of conditions $(q^i | i < \theta)$ and sets $w_i \subseteq \mathrm{Ord}$ such that

$$\mathrm{card}(w_i) < \kappa_{m-1}, \ q^i \leqslant p, \ q^i \Vdash \dot{h}(\check{i}) \in w_i \text{ and } q^i \upharpoonright m = p \upharpoonright m.$$

By the $< \kappa_{m-1}^+$-completeness of $(P_{\geqslant m}, \leqslant)$ take $q \in P$ such that $\forall i < \theta \ q \leqslant q^i$. Let $w = \bigcup_{i<\theta} w_i$. Since κ_{m-1} is regular we have $\mathrm{card}(w) \leqslant \kappa_{m-1}$ and $\sup(w) < \kappa_{m-1}$. Then $q \Vdash \mathrm{range}(\dot{h}) \subseteq w \subseteq \sup(w) < \kappa_{m-1}$, contradicting that p forces $\dot{h}$ to be onto κ_{m-1}. Thus κ_{m-1} is a cardinal in $V[G]$.

The argument for κ_{m-1}^+ being a cardinal is virtually the same. This proves (4). $\qquad \square$

(5) Let $m < \omega$. Then no ordinal $\xi \in (\kappa_{m-1}^+, \kappa_m)$ is a cardinal in $V[G]$. This follows immediately from the basic properties of a Levy collapse.

Hence we obtain the following cardinality pattern in $V[G]$:

(6) $\aleph_1^{V[G]} = \aleph_1^V$, $\aleph_2^{V[G]} = \aleph_2^V$, $\aleph_3^{V[G]} = \kappa_0$, $\aleph_4^{V[G]} = \kappa_0^+$, ..., $\aleph_{2n+3}^{V[G]} = \kappa_n$, $\aleph_{2n+4}^{V[G]} = \kappa_n^+$, ..., $\aleph_\omega^{V[G]} = \kappa$.

To prove $\mathrm{MS}(\aleph_3, \aleph_5, \aleph_7, \ldots ; \omega_1)$ in $V[G]$ consider, in $V[G]$, an $F : [\aleph_\omega]^{<\omega} \to \aleph_\omega$ and a sequence $(S_n)_{n<\omega}$ of sets $S_n \subseteq \mathrm{cof}_{\omega_1}$ stationary in κ_n. Take names $\dot{F}$ and $\dot{S}_n$ and a condition $p \in P$ such that

$$p \Vdash (\dot{F} : [\aleph_\omega]^{<\omega} \to \aleph_\omega \text{ and } \forall n < \omega \ (\dot{S}_n \subseteq \mathrm{cof}_{\omega_1} \wedge \dot{S}_n \text{ is stationary in } \kappa_n)).$$

It suffices to "densely show" the existence of a set $X \subseteq \aleph_\omega$ such that

$$\forall n < \omega \ \sup((F''[X]^{<\omega}) \cap \aleph_{2n+3}) \leqslant \sup(X \cap \aleph_{2n+3}) \in S_n.$$

We shall fix the values $\dot{F}(x)$ for a large set of $x \in [\kappa]^{<\omega}$. We shall define conditions $p(x) \in P$ such that $p(x) \Vdash \dot{F}(x) = \check{\alpha}$ for some ordinal $\alpha < \kappa$. The recursive definition of the function $p(x)$ will proceed along the well-ordering $<^*$ of $[\mathrm{Ord}]^{<\omega}$ by *largest differences*:

$$x <^* y \text{ iff } \max(x \triangle y) \in y.$$

Let t be a type (relative to the sequence $\kappa_0 < \kappa_1 < \ldots < \kappa$). A sequence $(p(x)|x \in [\kappa]^t)$ is called *partially constant* (*p.c.*) (below p) if the following two conditions are satisfied:

(7) $\forall x \in [\kappa]^t \ p(x) \leqslant p$;

(8) $\forall x, y \in [\kappa]^t \ \forall n < \omega \ (x \setminus \kappa_{n-1} = y \setminus \kappa_{n-1} \rightarrow p_n(x) = p_n(y))$.

By (8), the n-th component of $p(x)$ is determined by $x \setminus \kappa_{n-1}$. Later we shall use mutual indiscernibility to make $p_n(x)$ only dependent on $x \cap (\kappa_{n-1}, \kappa_n)$.

We shall construct a countable family of partially constant sequences which decide certain values of $\dot{F}$ and which witness some compatibilities. Note that the constant sequence $p(x) \equiv p$ is partially constant. The following two extension properties yield new partially constant sequences starting from given ones.

(9) Let $t, \bar{t}$ be types and $a \subseteq \omega$ such that for all $z \in [\kappa]^t$ type$(z \upharpoonright a) = \bar{t}$ holds. Let $(p(x)|x \in [\kappa]^{\bar{t}})$ be a p.c. sequence. Define $(q(z)|z \in [\kappa]^t)$ by $q_n(z) = p_n(z \upharpoonright a)$. Then $(q(z)|z \in [\kappa]^t)$ is p.c.

Proof. Let $x, y \in [\kappa]^t$ and $x \setminus \kappa_{n-1} = y \setminus \kappa_{n-1}$. Then $(x \upharpoonright a) \setminus \kappa_{n-1} = (y \upharpoonright a) \setminus \kappa_{n-1}$ and

$$q_n(x) = p_n(x \upharpoonright a) = p_n(y \upharpoonright a) = q_n(y),$$

which proves (9). $\qquad\square$

(10) Let t be a type and let $(p(x)|x \in [\kappa]^t)$ be a p.c. sequence. Then there is a p.c. sequence $(q(x)|x \in [\kappa]^t)$ and a sequence $(v(z)|z \in [\kappa]^t)$ of finite sets of ordinals $< \kappa$ such that

$$\forall x \in [\kappa]^t \ (q(x) \leqslant p(x) \text{ and } \forall z \subseteq x \ \exists \alpha \in v(x) \ q(x) \Vdash \dot{F}(z) = \check{\alpha}).$$

Proof. Define sequences $(r(x)|x \in [\kappa]^t)$ and $(v(x)|x \in [\kappa]^t)$ by simultaneous induction on $<^*$ such that the following two properties hold:

(11) $r(x) \leqslant p(x)$;

(12) $\forall u, x \in [\kappa]^t, u <^* x \ \forall n < \omega \ (u \setminus \kappa_{n-1} = x \setminus \kappa_{n-1} \rightarrow r_n(u) \geqslant r_n(x))$.

Let $x \in [\kappa]^t$ and let $r(u), v(u)$ be adequately defined for $u <^* x$, $x \in [\kappa]^t$. Then define $\bar{r} \in P$ by

$$\bar{r}_n = p_n(x) \cup \bigcup \{r_n(u)|u <^* x, u \setminus \kappa_{n-1} = x \setminus \kappa_{n-1}\}.$$

$\bar{r}$ is a properly defined condition in P by the growth condition (12) and the closure conditions of the Levy collapses. Also $\bar{r} \leqslant p(x)$. Then choose $r(x) \leqslant \bar{r}$ and $v(x) \in [\kappa]^{<\omega}$ such that

$$\forall z \subseteq x \ \exists \alpha \in v(x) \ r(x) \Vdash \dot{F}(z) = \check{\alpha}.$$

Note that the recursive condition (12) is satisfied at x: if $u \in [\kappa]^t, u <^* x$ and $u \setminus \kappa_{n-1} = x \setminus \kappa_{n-1}$, then $r_n(x) \leqslant \bar{r}_n \leqslant r_n(u)$.

Finally define the sequence $(q(x)|x \in [\kappa]^t)$ by

$$q_n(x) = p_n(x) \cup \bigcup \{r_n(u)|u \in [\kappa]^t, u \setminus \kappa_{n-1} = x \setminus \kappa_{n-1}\}.$$

$(q(x))$ is p.c. by construction and satisfies claim (10). $\qquad\square$

(13) Let $t, \bar{t}$ be types and $a_0, a_1 \subseteq \omega$ be such that for all $y \in [\kappa]^t$: $\text{type}(y \upharpoonright a_0) = \text{type}(y \upharpoonright a_1) = \bar{t}$. Let $(p(x)|x \in [\kappa]^{\bar{t}})$ be a p.c. sequence. Then there is a p.c. sequence $(q(y)|y \in [\kappa]^t)$ such that for all $y \in [\kappa]^t$, if $p(y \upharpoonright a_0)$ and $p(y \upharpoonright a_1)$ are compatible in P, then

$$q(y) \leqslant p(y \upharpoonright a_0) \text{ and } q(y) \leqslant p(y \upharpoonright a_1).$$

Proof. For $y \in [\kappa]^t$ define $q(y)$ by $q_n(y) = p_n(y \upharpoonright a_0) \cup p_n(y \upharpoonright a_1)$ if the right-hand side is a condition in $\text{Col}(\kappa_{n-1}^+, < \kappa_n)$, and $q_n(y) = p_n(y \upharpoonright a_0)$ otherwise.

We only have to check that (q) is p.c. Let $x, y \in [\kappa]^t$ and $x \setminus \kappa_{n-1} = y \setminus \kappa_{n-1}$. By definition

$$q(x) \leqslant p(x \upharpoonright a_0) \leqslant p.$$

$x \setminus \kappa_{n-1} = y \setminus \kappa_{n-1}$ implies that $(x \upharpoonright a_0) \setminus \kappa_{n-1} = (y \upharpoonright a_0) \setminus \kappa_{n-1}$ and $(x \upharpoonright a_1) \setminus \kappa_{n-1} = (y \upharpoonright a_1) \setminus \kappa_{n-1}$. Since (p) is p.c., $p_n(x \upharpoonright a_0) = p_n(y \upharpoonright a_0)$ and $p_n(x \upharpoonright a_1) = p_n(y \upharpoonright a_1)$.

Case 1. $p_n(x \upharpoonright a_0), p_n(x \upharpoonright a_1)$ are compatible in $\text{Col}(\kappa_{i-1}^+, < \kappa_i)$. Then

$$q_n(x) = p_n(x \upharpoonright a_0) \cup p_n(x \upharpoonright a_1) = p_n(y \upharpoonright a_0) \cup p_n(y \upharpoonright a_1) = q_n(y).$$

Case 2. $p_n(x \upharpoonright a_0), p_n(x \upharpoonright a_1)$ are incompatible in $\text{Col}(\kappa_{i-1}^+, < \kappa_i)$. Then

$$q_n(x) = p_n(x \upharpoonright a_0) = p_n(y \upharpoonright a_0) = q_n(y).$$

This proves (13).   □

Properties (9), (10) and (13) are construction principles for p.c. sequences $q(x)$ from p.c. sequences $p(x)$. By an appropriate organisation one can construct a sequence of p.c. sequences

$$p^0, p^1, v^1, p^2, v^2, \dots$$

such that p^0 is the constant sequence $p^0(x) \equiv p$ and such that for any given p^k and any constellation of types as in (9), (10) or (13) there are sequences p^l and possibly v^l which relate to p^k as q and v relate to p in (9), (10) or (13) respectively. We may also assume that every p^k, v^k is definable over some common sufficiently large structure

$$(V_\theta, \in, \dots, \dot{F}, \dot{S}_0, \dot{S}_1, \dots, p), \text{ where } \theta \supseteq \kappa.$$

By the mutual Ramseyness of $\kappa_0, \kappa_1, \dots$ take a mutually homogeneous system $I_0, I_1, \dots$ for the structure $(V_\theta, \in, \dots, \dot{F}, \dot{S}_0, \dot{S}_1, \dots, p)$ such that $I_i \subseteq \kappa_i$ and such that $\text{card}(I_i) = \kappa_i$. One can check that each of these indiscernibles is inaccessible.

Consider any sequence $I'_0, I'_1, \dots$ such that each I'_i is a subset of I_i of order type ω_1. We shall later choose such a sequence appropriately for the mutual stationarity property. We view $I' = \bigcup_{i < \omega} I'_i$ as a system of generating indiscernibles for a substructure of $(V_\theta, \in, \dots, \dot{F}, \dot{S}_0, \dot{S}_1, \dots, p)$. Let

$$P' = \{p^k(z)|z \subseteq I', k < \omega\} \subseteq P.$$

We intend to apply $\text{MA}_{\aleph_1}$ to P' to produce a sufficiently generic condition in P. This requires the countable chain condition and certain densities.

(14) $(P', \leqslant)$ satisfies the countable chain condition.

Proof. Let $A \subseteq P'$ be uncountable. By possibly thinning out A we may assume that there is some $k < \omega$ such that every $q \in A$ is of the form $q = p^k(z)$ for some $z \subseteq I'$, i.e.,

$$A = \{p^k(z)|z \in Z\},$$

where $Z \subseteq [I']^t$ is uncountable and t is the appropriate argument type for p^k. By a Δ-system argument one can assume that Z is an uncountable Δ-system of finite sets with kernel a:

$$\forall z \in Z : a \subseteq z \text{ and } \forall z, z' \in Z, z \neq z' : (z \setminus a) \cap (z' \setminus a) = \emptyset.$$

Since the set $\bigcup_{n<\omega}[I' \cap (\kappa_{n-1}, \max(a \cap (\kappa_{n-1}, \kappa_n)))]$ is countable there is an uncountable $Z' \subseteq Z$ such that

$$\forall z \in Z' \ \forall n < \omega : \max(a \cap (\kappa_{n-1}, \kappa_n)) < \min((z \setminus a) \cap (\kappa_{n-1}, \kappa_n)).$$

Choose $x \in Z'$. Again, there is an uncountable $Z'' \subseteq Z'$ such that

$$\forall z \in Z'' \ \forall n < \omega : \max(x \cap (\kappa_{n-1}, \kappa_n)) < \min((z \setminus a) \cap (\kappa_{n-1}, \kappa_n)).$$

Choose $y \in Z''$. We shall show that $p^k(x)$ and $p^k(y)$ are compatible in P. Let $n < \omega$. It suffices to show that $p_n^k(x)$ and $p_n^k(y)$ are compatible in $\mathrm{Col}(\kappa_{n-1}^+, < \kappa_n)$. Since p^k is p.c. we have

$$p_n^k(y) = p_n^k((x \cap \kappa_{n-1}) \cup (y \cap (\kappa_{n-1}, \kappa_n))),$$

i.e., we can assume for the compatibility argument that $x \cap \kappa_{n-1} = y \cap \kappa_{n-1}$.

Case 1. $(x \setminus a) \cap (\kappa_{n-1}, \kappa_n) = \emptyset$, i.e., $x \cap (\kappa_{n-1}, \kappa_n) = a \cap (\kappa_{n-1}, \kappa_n)$. Since x and y have the same type, also $y \cap (\kappa_{n-1}, \kappa_n) = a \cap (\kappa_{n-1}, \kappa_n)$. The term $p_n^k(x)$ is an element of V_{κ_n} and can thus be viewed as a term of the structure $(V_\theta, \in, \ldots, \dot{F}, \dot{S}_0, \dot{S}_1, \ldots, p)$ mapping x to an ordinal $< \kappa_n$. Similarly, $p_n^k(y)$ can be viewed as the same term mapping y to an ordinal $< \kappa_n$. Since $x \cap \kappa_n = y \cap \kappa_n$ the mutual homogeneity of the system I implies that

$$p_n^k(x) = p_n^k(y);$$

hence both sides are trivially compatible in $\mathrm{Col}(\kappa_{n-1}^+, < \kappa_n)$.

Case 2. $(x \setminus a) \cap (\kappa_{n-1}, \kappa_n) \neq \emptyset$. Then let $\beta = \min((x \setminus a) \cap (\kappa_{n-1}, \kappa_n))$ and $\gamma = \min((y \setminus a) \cap (\kappa_{n-1}, \kappa_n))$. As in *Case 1*, the terms $p_n^k(x) \upharpoonright (\kappa_{n-1}^+ \times \beta)$ and $p_n^k(y) \upharpoonright (\kappa_{n-1}^+ \times \gamma)$ can be viewed as ordinals $< \beta$ and $< \gamma$ respectively. Since $x \cap \beta = y \cap \beta = y \cap \gamma$ the mutual homogeneity of the sytem I implies

$$p_n^k(x) \upharpoonright (\kappa_{n-1}^+ \times \beta) = p_n^k(y) \upharpoonright (\kappa_{n-1}^+ \times \gamma)$$

and

$$\mathrm{dom}(p_n^k(x)) \subseteq \kappa_{n-1}^+ \times \gamma.$$

Hence $p_n^k(x)$ and $p_n^k(y)$ are compatible in $\mathrm{Col}(\kappa_{n-1}^+, < \kappa_n)$.

So in both cases, we obtain compatibility of $p_n^k(x)$ and $p_n^k(y)$ for all $n < \omega$; hence $p^k(x)$ and $p^k(y)$ are compatible in P. By construction of the sequence $p^0, p^1, v^1, p^2, v^2, \ldots$ P' contains a compatibility element for $p^k(x)$ and $p^k(y)$, and so $p^k(x)$ and $p^k(y)$ are compatible in P'. Thus no uncountable subset of P' is an antichain in P', which proves (14). $\qquad\square$

Let $W = \bigcup\{v^k(z) | z \subseteq I', k < \omega\}$ be the set of possible values $\dot{F}(z)$ on the indiscernibles I'.

(15) Let $y \subseteq I'$, y finite. Then the set

$$D_y = \{r \in P' | \exists \alpha \in W \ r \Vdash_P \dot{F}(y) = \check{\alpha}\}$$

is dense in P'. Observe that the forcing $\Vdash_P$ in the definition of D_y is done with respect to the full partial order P.

Proof. Let $p^k(x) \in P'$, $x \subseteq I'$. Let $z = x \cup y \subseteq I'$. By the construction of the sequence $p^0, p^1, v^1, p^2, v^2, \ldots$ according to property (9) there is $l < \omega$ such that $p^k(x) = p^l(z)$. Furthermore, according to (9) there is some $m < \omega$ and some finite $z \subseteq I'$ such that $x, y \subseteq z$ and

$$p^l(z) \leqslant p^k(x) \text{ and } p^l(z) \Vdash \dot{F}(y) = v^l\check{(z)} \in \check{W}.$$

This proves (15).   □

By $\mathrm{MA}_{\aleph_1}$ take a generic filter $H' \subseteq P'$ for the family $\{D_y | y \subseteq I', y \text{ finite}\}$. Define a condition $p' \in P$ by $p'_n = \bigcup\{r_n | r \in H'\}$. Observe that all r_n for $r \in H'$ are compatible by the genericity of H'. Also $p' \leqslant p$.

(16) $p' \Vdash \dot{F}''[I']^{<\omega} \subseteq \check{W}$.

Proof. Let $y \in I'$, y finite. By the genericity of H' there is $r \in H' \cap D_y$. Take $\alpha \in W$ such that $r \Vdash \dot{F}(y) = \check{\alpha}$. Then $r \Vdash \dot{F}(y) \in \check{W}$ and $p' \leqslant r$. This proves (16).   □

(17) $\forall n < \omega, \sup(W \cap \kappa_n) \leqslant \sup I'_n$.

Proof. Let $\alpha = v^k(z) \in W \cap \kappa_n$, $z \in I'$. Since I_n is cofinal in κ_n there is some $\zeta \in I_n$ such that $\alpha < \zeta < \kappa_n$. Since I'_n has limit order type there is $\zeta' \in I'_n$ such that $\max(z \cap \kappa_n) < \zeta'$. By the indiscernibility properties $\alpha = v^k(z) < \zeta$ implies $\alpha = v^k(z) < \zeta' < \sup I'_n$, proving (17).   □

In the following we shall isomorphically shift the indiscernible sets $I'_0, I'_1, \ldots$ to some other such sets $I''_0, I''_1, \ldots$ so that the stationarity property $\sup I''_n \in \dot{S}_n$ is also forced. If $I''_0, I''_1, \ldots$ is another sequence such that each I''_i is a subset of I_i of order type ω_1, set $I'' = \bigcup_{i<\omega} I''_i$. These sets of generating indiscernibles are order-isomorphic by some isomorphism $\pi : I' \cong I''$. The isomorphism can canonically be extended to an isomorphism $\tilde{\pi}$ of the substructures of $(V_\theta, \in, \ldots, \dot{F}, \dot{S}_0, \dot{S}_1, \ldots, p)$ generated by the indiscernible sets I' and I'' respectively: $\tilde{\pi}$ is defined by $t(z) \mapsto t(\pi''z)$ for Skolem terms t of $(V_\theta, \in, \ldots, \dot{F}, \dot{S}_0, \dot{S}_1, \ldots, p)$ and $z \subseteq I'$. In particular, $\tilde{\pi}$ preserves all components of the previous construction.

Let $P'' = \{p^k(z) | z \subseteq I'', k < \omega\} \subseteq P$. Then $\tilde{\pi} \restriction P' : (P', \leqslant) \cong (P'', \leqslant)$ is a well-defined isomorphism. $\tilde{\pi} \restriction W : W \cong W'' = \{v^k(z) | z \subseteq I'', k < \omega\}$ takes the sets of possible values into each other. For each $y \subseteq I'$, y finite, the set $D''_y = \tilde{\pi}''D_y$ satisfies

$$D''_y = \{r \in P'' | \exists \alpha \in W'' \; r \Vdash_P \dot{F}(y) = \check{a}\}.$$

The set $H'' = \tilde{\pi}''H'$ is generic in P'' for the family $\{D''_y | y \subseteq I'', y \text{ finite}\}$. Then the condition p'' defined by $p''_n = \bigcup\{r_n | r \in H''\}$ satisfies the analogue of (16): $p'' \Vdash \dot{F}''[I'']^{<\omega} \subseteq \check{W}''$. Finally the analogue of (17) holds: $\forall n < \omega \; \sup(W'' \cap \kappa_n) \leqslant \sup I''_n$.

We show that the domains of the condition p' are closely positioned around the generating indiscernibles in I'.

(18) Let $r = p^k(z)$, $z \subseteq I'$, and $n < \omega$. Let $\xi \in I_n \setminus I'_n$. Then

$$\mathrm{dom}\, p^k_n(z) \cap \kappa^+_{n-1} \times [\xi, \min(I_n \setminus (\xi+1))) = \emptyset.$$

Proof. Assume not. Let $\xi' = \min(I_n \setminus (\xi+1))$ and

$$s(z, \xi') = \sup\{\beta < \xi' | \exists \alpha < \kappa^+_{n-1}(\alpha, \beta) \in \mathrm{dom}\, p^k_n(z)\}.$$

Since $\operatorname{card}(p_n^k) \leqslant \kappa_{n-1}$ and by the inaccessibility of the indiscernibles,

$$\xi \leqslant s(z, \xi') < \xi'.$$

Let $\xi'' = \min(I_n \setminus (\xi' + 1))$. Let $z'' \subseteq I_n$ with $\operatorname{type}(z'') = \operatorname{type}(z)$, $z'' \cap \xi' = z \cap \xi'$ and

$$\xi' \in z \to \xi'' = \min(z'' \setminus \xi') \text{ and } \xi' \notin z \to \xi'' < \min(z'' \setminus \xi').$$

By indiscernibility, $s(z'', \xi'') = s(z, \xi')$ and so $s(z'', \xi'') < \xi'$. Again by indiscernibility, observing that $\xi \notin z$: $s(z, \xi') < \xi$. But this contradicts $\xi \leqslant s(z, \xi')$, proving (18). $\qquad\square$

Since the condition p' is a union of conditions $p_n^k(z)$ as in (18) it follows that:
(19) Let $n < \omega$ and $\xi \in I_n \setminus I_n'$. Then

$$\operatorname{dom} p_n' \cap \kappa_{n-1}^+ \times [\xi, \min(I_n \setminus (\xi + 1))) = \emptyset$$

and hence

$$\operatorname{dom} p_n' \subseteq \kappa_{n-1}^+ \times (\min I_n \cup \bigcup_{\xi \in I_n'} [\xi, \min(I_n \setminus (\xi + 1)))).$$

Define $\bar{p} \in P$ by

$$\bar{p}_n = p_n' \restriction (\kappa_{n-1}^+ \times \min I_n).$$

Since the elements of I' are indiscernible with respect to the constant p we have that p_n is "below" the minimum of I_n. Hence $\bar{p} \leqslant p$. The above isomorphism $\pi : I' \cong I''$ of indiscernibles also shows that for any I'' as above we obtain:
(20) $\bar{p}_n = p_n'' \restriction (\kappa_{n-1}^+ \times \min I_n)$.

Consider a P-generic filter $\bar{G} \ni \bar{p}$ over V. In $V[\bar{G}]$, $\bar{S}_n = \dot{S}_n^{\bar{G}} \subseteq \kappa_n$ is stationary in κ_n and $\bar{S}_n \subseteq \operatorname{cof}_{\omega_1}$. We need elements of $\bar{S}_n$ which are high-level limits of indiscernibles. It is easy to see that:
(21) For all $n < \omega$ the set $I_n^* = \{\beta < \kappa_n | \forall \delta < \beta, \operatorname{otp}(I_n \cap (\delta, \beta)) > \kappa_{n-1}^+\}$ is closed unbounded in κ_n.

Take $\beta_n \in I_n^* \cap \bar{S}_n$. Since the forcing P is $< \aleph_2$-complete, $(\beta_n | n < \omega) \in V$ and every β_n has cofinality ω_1 in V. Let $q \leqslant \bar{p}$ such that $q \Vdash \forall n < \omega \beta_n \in \dot{S}_n$.

The following argument takes place in the ground model V. For each $n < \omega$ choose a strictly increasing sequence $(\beta_n^i | i < \omega_1)$ converging to β_n such that

$$\forall i < \omega_1 \operatorname{card}(I_n \cap (\beta_n^i, \beta_n^{i+1})) \geqslant \kappa_{n-1}^+.$$

Since $\operatorname{card}(q_n) < \kappa_{n-1}^+$ there exists $\xi_n^i \in I_n \cap (\beta_n^i, \beta_n^{i+1})$ such that:
(22) $\operatorname{dom} q_n \cap \kappa_{n-1}^+ \times [\xi_n^i, \min(I_n \setminus (\xi_n^i + 1))) = \emptyset$.

We can now define appropriate sets I_n'' of indiscernibles for the mutual stationarity property. Let $I_n'' = \{\xi_n^i | i < \omega_1\}$ and $I'' = \bigcup_{n < \omega} I_n''$. Define p'' and W'' from I'' as above. Then

$$p'' \Vdash \dot{F}''[I'']^{<\omega} \subseteq \check{W}''.$$

(23) The conditions q and p'' are compatible in P.

Proof. Let $n < \omega$. By (19) transferred to I'' we have

$$\operatorname{dom} p_n'' \subseteq \kappa_{n-1}^+ \times (\min I_n \cup \bigcup_{\xi \in I_n''} [\xi, \min(I_n \setminus (\xi + 1)))).$$

By (20), $p_n'' \restriction (\kappa_{n-1}^+ \times \min I_n) = \bar{p}_n \geqslant q_n$. Hence q and p'' are compatible on the rectangle $\kappa_{n-1}^+ \times \min I_n$. By (22), the other rectangles $\kappa_{n-1}^+ \times [\xi_n^i, \min(I_n \setminus (\xi_n^i + 1)))$

where p_n'' might be defined do not intersect the domain of q_n. So there is no conflict between q_n and p_n''. This proves (23). ◻

Let $r \leqslant q, p''$. By the analogues of (16) and (17) for I'',

$$r \Vdash \forall n < \omega \ \sup((\dot{F}''[I'']^{<\omega}) \cap \aleph_{2n+3}) \leqslant \sup(I'' \cap \aleph_{2n+3}) = \check{\beta}_n \in \dot{S}_n.$$

So one can densely force the mutual stationarity of $(\dot{S}_n | n < \omega)$, i.e., the mutual stationarity property $\mathrm{MS}(\aleph_3, \aleph_5, \aleph_7, \ldots; \omega_1)$ holds in $V[G]$. ◻

3. Variants and discussion

The principle $\mathrm{MS}(\ldots)$ allows many variations of parameters. We shall discuss some typical instances which are also true in the above or similar forcing extensions.

In the construction a mutual stationarity substructure of size $\aleph_1$ is constructed from $\aleph_1$ indiscernibles using $\mathrm{MA}_{\aleph_1}$. The indiscernibles, among other things, ensure that the supremum of the substructure below certain cardinals lies in a given stationary set. The point of the previous construction was that this is possible if the stationary set consists of ordinals of cofinality ω_1 but the argument goes through *a fortiori* for ordinals of countable cofinality. So one may freely alternate between the cofinalities ω_1 and ω to obtain

Theorem 3.1. *Let κ be a measurable cardinal. Then in the forcing extension of the main Theorem 1.6, $\mathrm{MS}(\aleph_3, \aleph_5, \aleph_7, \ldots; (\lambda_n)_{n<\omega})$ holds for every sequence $(\lambda_n)_{n<\omega}$ with $\forall n < \omega \lambda_n \in \{\omega, \omega_1\}$. E.g., $\mathrm{MS}(\aleph_3, \aleph_5, \aleph_7, \aleph_9, \ldots; \omega_1, \omega_0, \omega_1, \omega_0, \ldots)$ is true in the extension.*

For the proof of Theorem 1.7 consider the suprema $\gamma_n = \sup((F''[I'']^{<\omega}) \cap \aleph_{2n+2})$ of the mutual stationary substructure below the cardinals $\aleph_2, \aleph_4, \ldots$. Since $\aleph_{2n+2}$ is a regular cardinal we may assume that the structure

$$(V_\theta, \in, \ldots, \dot{F}, \dot{S}_0, \dot{S}_1, \ldots, p)$$

contains functions w^l which code suprema of the value function v^k below $\aleph_{2n+2}$: if $\mathrm{dom}(v^k) = [\kappa]^t$, then

$$(3.1) \qquad \forall z \in [\kappa]^t \ (v^k(z) < \aleph_{2n+2} \rightarrow v^k(z) < w^l(z \setminus \aleph_{2n+1})).$$

We claim that then $\mathrm{cof}(\gamma_n) = \omega$. By claim (15) of the main proof, every element $\xi \in F''[I'']^{<\omega}$ is of the form $\xi = v^k(z)$ for some $z \subseteq I''$ and $k < \omega$. By (3.1), $\xi = v^k(z) < w^l(z \setminus \aleph_{2n+1})$. Since $z \setminus \aleph_{2n+1}$ are indiscernibles for values $< \aleph_{2n+2}$ the countable set

$$\{w^l(x) | x \subseteq I'' \setminus \aleph_{2n+2}\}$$

bounds every element of $(F''[I'']^{<\omega}) \cap \aleph_{2n+2}$. On the other hand, we may also assume that all the values in $\{w^l(x) | x \subseteq I'' \setminus \aleph_{2n+2}\}$ are realised as values in $F''[I'']^{<\omega}$. Hence $\mathrm{cof}(\gamma_n) = \omega$. This means that in the generic extension we can find mutually generic substructures whose suprema below the even $\aleph_{2n+2}$ have cofinality ω and below the odd $\aleph_{2n+3}$ have cofinality ω_1. This proves Theorem 1.7.

In view of this theorem it is natural to ask whether even the mutual stationarity property $\mathrm{MS}(\aleph_2, \aleph_3, \aleph_4, \aleph_5, \ldots; \omega_0, \omega_1, \omega_0, \omega_1, \ldots)$ holds in the model. By the methods of [5] one can however show that the consistency strength of that property is strictly larger than the existence of one measurable cardinal.

The combinatorics of our forcing construction will also go through for product forcings of the form $Q = \prod_{n<\omega} \mathrm{Col}(\lambda_n, < \kappa_n)$ where each λ_n is a regular cardinal in

the interval $[\kappa_{n-1}^+, \kappa_n)$. So from one measurable cardinal, one can force properties such as

$$\mathrm{MS}(\aleph_4, \aleph_9, \aleph_{16}, \aleph_{25}, \ldots; \omega_1) \text{ or } \mathrm{MS}(\aleph_{\omega+2}, \aleph_{\omega \cdot 2+2}, \aleph_{\omega \cdot 3+2}, \ldots; \omega_1).$$

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