A construction of linear filtrations for bundles on $\mathbf{P}^1_{\mathbb{Z}}$

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Abstract

We give an algorithm for constructing a filtration of any vector bundle with rank 2 on $\mathbf{P}^1_A$, where A is an Euclidean domain. All the quotients of this filtration are linear bundles. In other words, the algorithm takes any invertible 2-matrix σ over $A[x, x^{-1}]$, and gives 2-matrices λ over $A[x]$ and ρ over $A[x^{-1}]$ such that the product $\lambda\sigma\rho$ is an upper triangular matrix.

Keywords: vector bundle, arithmetic surface, projective line, filtration, Euclidean domain, reduction, upper triangular matrix.

Introduction

We study vector bundles on the arithmetic surface $\mathbf{P}^1_A$, where A is a Dedekind domain. In the case when A is a Euclidean domain, we give an algorithm for constructing a linear filtration for the rank 2 vector bundles. Our construction uses the ideas of Hanna's proof of the existence of such filtrations; see [1].

The study of vector bundles over arithmetic surfaces, for instance on $\mathbf{P}^1_{\mathbb{Z}}$, may be regarded as a synthesis of algebraic theory of vector bundles and theory of vector bundles over compactified curves. Our work is one of the steps toward studying vector bundles on the compactification of $\mathbf{P}^1_{\mathbb{Z}}$.

It would be interesting to obtain an interpretation of the results in the spirit of Parshin's works on higher Bruhat–Tits trees and vector bundles over algebraic surfaces (see, for example, [2]). In the one-dimensional case, this approach is due to Serre (see [4]).

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1 General considerations

We shall study vector bundles over $\mathbf{P}^1_A$ for a Dedekind domain A, in particular for $A = \mathbb{Z}$. It is instructive and more natural to state a few results in greater generality. Thus let A be a Noetherian commutative ring.

We write usually $\mathcal{O}$ and $\mathcal{O}(d)$ instead of $\mathcal{O}_X$ and $\mathcal{O}_X(d)$ for a suitable scheme X, especially if X can be easily specified. As usual, let

$$\mathbf{P}^1_A = \operatorname{Proj} A[t_0, t_1], \quad \deg t_0 = \deg t_1 = 1.$$

In addition, $\mathcal{O}(U_0) = A[x]$, $\mathcal{O}(U_1) = A[x^{-1}]$, and $\mathcal{O}(U_{01}) = A[x, x^{-1}]$, where $x = t_1/t_0$, U_i denotes the complement to the zero locus of t_i, and $U_{01} = U_0 \cap U_1$.

1.1 Basic results

We shall start with a brief review of vector bundles on $\mathbf{P}^1_A$.

1.1.1 Theorem (Grothendieck, [6])**.** *Let F be a field. Any vector bundle on $\mathbf{P}^1_F$ is isomorphic to a sum of line bundles with uniquely defined summands.*

Line bundles can be described as follows:

1.1.2 Theorem (EGA, [?])**.** *Any line bundle on $\mathbf{P}^n_A$ is isomorphic to a bundle of the form $p^*L \otimes \mathcal{O}(d)$, where L is a line bundle on* $\operatorname{Spec} A$ *and $p : \mathbf{P}^n_A \to \operatorname{Spec} A$ is a structure morphism.*

In particular, any line bundle on $\mathbf{P}^1_F$ is isomorphic to $\mathcal{O}(d)$ for some $d \in \mathbb{Z}$.

Note that the situation for $A = \mathbb{Z}$ is more complicated. More generally, for certain Dedekind domains A, there exist indecomposable rank 2 vector bundles on $\mathbf{P}^1_A$ (see [3], [8]). It is an open question whether any vector bundle on $\mathbf{P}^1_A$ for a Dedekind ring A admits a filtration with linear bundles as quotients. Let us cite a few known results in this direction.

1.1.3 Theorem (Hanna, [1])**.** *Let A be a PID, and let E be any vector bundle on $\mathbf{P}^1_A$. Then E has a filtration*

$$0 \subseteq E_0 \subseteq E_1 \subseteq \ldots \subseteq E_t = E$$

such that E_i/E_{i-1} is a line bundle when $i < t$ and E_t/E_{t-1} has rank at most two.

The question is answered completely for any Euclidean ring.

1.1.4 Theorem (Hanna, [1])**.** *Let A be a Euclidean domain, and let F be any vector bundle on $\mathbf{P}^1_A$. Then F has a filtration*

$$0 = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n = F$$

such that F_i/F_{i-1} is a line bundle ($1 \leq i \leq n = \operatorname{rk} F$).

In particular, every bundle on $\mathbf{P}^1_{\mathbb{Z}}$ admits a filtration with linear bundles as a quotients.

1.2 Gluing

Let A be a PID, and let $\sigma \in \mathrm{GL}_n(A[x, x^{-1}])$. To σ one associates a vector bundle on $\mathbf{P}^1_A$ as follows: $E\big|_{U_0} = \mathcal{O}e_1 + \cdots + \mathcal{O}e_r$, $E\big|_{U_1} = \mathcal{O}f_1 + \cdots + \mathcal{O}f_r$, and

$$[e_1, \ldots, e_r]\sigma = [f_1, \ldots, f_r] \tag{1}$$

over U_{01}, so that $f_j = \sum_{i=1}^{r} \sigma_{i,j} e_i$.

By the Quillen–Suslin Theorem, every finitely generated projective A-module is free, thus any vector bundle of rank r on $\mathbf{P}^1_A$ can be obtained in that way. In this case, the isomorphism class of such bundle is an element of the double quotient

$$\mathrm{Vect}_r(\mathbf{P}^1) = \mathrm{GL}_r(A[x]) \diagdown \mathrm{GL}_r(A[x, x^{-1}]) \diagup \mathrm{GL}_r(A[x^{-1}]). \tag{2}$$

1.3 Notation

1.3.1. Let F be a functor from the category of A-algebras. We set

$$F_{xy} = F(A[x, y]), \quad F_x = F(A[x]), \text{ and } \quad F_y = F(A[y]).$$

Throughout this paper, we will use the following notation:

$G = \mathrm{GL}_r$	The group of invertible $r \times r$ matrices
$M = M_{r,r}$	The algebra of $r \times r$ matrices
T	The set of diagonal matrices in G
B	The standard Borel subgroup (upper-triangular matrices) in G

Let T be a set of variables. A row vector over $A[T]$ is said to be a T-*row*. A column vector over $A[T]$ is called T-*column*. Moreover, a T-row (T-column) is said to be a T-*unimodular* row (T-column) if it is unimodular over $A[T]$.

1.3.2. Let $\omega = (\omega_{i,j})$ be an $n \times n$ matrix, where $n \geq 1$. By $\theta_{i,j}$ we denote the cofactor of $\omega_{i,j}$. Set $\theta = (\theta_{i,j})$. Then we define θ^t, the transpose of θ, to be the adjugate of ω, and we denote it by ω^*.

The product of ω with its adjugate yields a diagonal matrix whose diagonal entries are $\det(\omega)$. More explicitly,

$$\omega\omega^* = \det(\omega) I_n,$$

where I_n is the $n \times n$ identity matrix.

1.3.3. Normalization. Let α be a nonzero xy-row and let α/x^* be an x-row defined by

$$\alpha/x^* = \alpha/x^n, \tag{3}$$

where n is the largest integer such that α/x^n is an x-row.

In this situation, we say that α/x^* is an *x-normalization* of α. We define a y-normalization map $\alpha \mapsto \alpha/y^*$ likewise.

Let $\omega \in M_{xy}$ be a matrix with at least one nonzero element in every row.

Then set

$$\omega/x^* = (\omega_{i,*}/x^*), \quad \text{and} \quad \omega/y^* = (\omega_{i,*}/y^*), \quad i = 1, \dots, r\,, \tag{4}$$

where $\omega_{i,*}$ is the ith row of ω.

Suppose $A = \mathbb{Z}$. For an example of an x-normalization, consider the matrix

$$\omega = \begin{bmatrix} 2y^3 & 3x + 4y^2 \\ 5y^4 & 6x^5 \end{bmatrix}.$$

Then we have

$$\omega/x^* = \begin{bmatrix} 2 & 3x^4 + 4x \\ 5 & 6x^9 \end{bmatrix}, \qquad \text{and} \qquad \omega/y^* = \begin{bmatrix} 2y^4 & 3 + 4y^3 \\ 5y^9 & 6 \end{bmatrix}.$$

1.3.4. Let K be the fraction field of A. By v_x and v_y we denote valuations

$$v_x,\ v_y : K(\mathbf{P}^1_K)^* \to \mathbb{Z}$$

corresponding to the points $x = 0 \in \mathbf{P}^1_K$ and $y = 0 \in \mathbf{P}^1_K$, respectively.

For instance, we have $v_x(x^2 + x^3) = 2$, and $v_y(x^2 + x^3) = -3$.

1.4 Euclidean rings

In this section, we recall briefly some basic facts about Euclidean rings (see [**?**]).

1.4.1. Definitions and notational conventions. A domain A is called Euclidean if there exists a map $\mathrm{ht} : A \to \{0,\ 1,\ \dots\}$ satisfying the following conditions:

(i) $\mathrm{ht}(a) = 0$ iff $a = 0$.

(ii) If $a, b \in A - \{0\}$, then there exist $q, r \in A$ such that $a = bq + r$, where $\mathrm{ht}(r) < \mathrm{ht}(b)$.

(iii) If b divides $a \neq 0$ then $\mathrm{ht}(a) \geq \mathrm{ht}(b)$.

(iv) $\mathrm{ht}(1) = 1$.

For instance, the ring $\mathbb{Z}$ becomes Euclidean if one defines $\mathrm{ht}(a) = |a|$. Another important example of a Euclidean domain is the ring of polynomials $F[x]$ over a field F. In this case, we define $\mathrm{ht}(f(x)) = 2^{\deg f(x)}$ (where it is understood that $\deg 0 = -\infty$).

Given a Euclidean domain A, we will assume that ht function is fixed .

1.4.2. $[a/b]$ **operation**. Let A be a Euclidean domain. By the above definition, if $a, b \in A - \{0\}$ then there exists $q \in A$ such that $a = bq + r$, where $\mathrm{ht}(a - bq) < \mathrm{ht}(b)$.

Define $[a/b]$ by the formula

$$[a/b] = q.$$

Of course, q is not uniquely determined in general.

1.4.3. Height of a row. Given an ideal I in a Euclidean domain A, we set

$$\mathrm{ht}(I) = \mathrm{ht}(a),$$

where $(a) = I$. It follows easily that the function ht is well-defined.

Let $(a_1, \dots, a_n)$ be a row vector over A. We set

$$\mathrm{ht}(a_1, \dots, a_n) = \mathrm{ht}(I),$$

where I is the ideal generated by the $a_1, \dots, a_n$.

1.4.4. Row reduction. Let $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$ be non-zero row vectors over A. Suppose that a and b are linearly dependent, i.e. $a_i b_j = a_j b_i$, $(i, j = 1, \dots, n)$. Then $[a/b]$ is defined by the formula

$$[a/b] := [u/v],$$

where u and v are the generators of the ideals $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$, respectively, such that

$$a_i v = u b_i, \quad (i = 1, \dots, n). \tag{5}$$

Notice that the definition of $[a/b]$ has an arbitrariness caused by the arbitrariness of the operation $[u/v]$ (see 1.4.2). However, $[a/b]$ is actually independent of the choice of u and v satisfying the condition (5).

In terms of $[a/b]$ one defines a row vector $r = (r_1, \dots, r_n)$ by the formula:

$$(r_1, \dots, r_n) := (a_1, \dots, a_n) - [a/b](b_1, \dots, b_n).$$

We have

$$\mathrm{ht}(r_1, \dots, r_n) < \mathrm{ht}(b_1, \dots, b_n).$$

2 Reduction algorithm

Throughout the remainder of this paper, A is a Euclidean domain (see 1.4) and

$$r = 2,$$

where $r = \mathrm{rk}(E)$, and E is a vector bundle on $\mathbf{P}^1_A$.

We shall construct an algorithm for reducing invertible matrices over $A[x, x^{-1}]$ to upper triangular form. In other words, for any E we shall construct an exact sequence:

$$0 \to L_1 \to E \to L_2 \to 0, \tag{6}$$

where the L_i are linear bundles on $\mathbf{P}^1_A$.

Geometrically, we need to find a linear bundle L and a map $L \to E$, which is nowhere zero. In other words, this is equivalent to showing that the projection $\mathbf{P}(E) \to \mathbf{P}^1_A$ admits a section, where $\mathbf{P}(E)$ is the projectivization of E.

Let σ be a matrix in G_{xy}. The algorithm takes σ, and gives matrices $\lambda \in G_x$ and $\rho \in G_y$ such that their product $\lambda\sigma\rho$ is an upper triangular matrix, i.e.

$$\lambda\sigma\rho = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix},$$

where by $*$ we denoted elements of $A[x, y]$. In the notation of 1.3.1, we write

$$\lambda\sigma\rho \in B_{xy}. \tag{7}$$

2.1 Algorithm description

Our goal in this section is to introduce x-reduction and y-reduction routines. The reduction to upper triangular form can be done by performing the above mentioned routines, one after the other.

The x-reduction routine takes $\sigma \in G_{xy}$ as input and gives a matrix $\lambda \in G_x$ such that

$$\lambda\sigma \text{ is } y\text{-special (see the definition in 2.2.1).} \tag{8}$$

The y-reduction routine takes matrices $\sigma \in G_{xy}$ and $\lambda \in G_x$ satisfying (8), and gives a matrix $\rho \in G_y$ such that the equality (7) holds.

Thus in order to reduce $\sigma \in G_{xy}$ we have to apply the x-reduction to σ, this step gives an input for the y-reduction. Applying the y-reduction to σ and $\lambda \in G_x$, we obtain the desired decomposition (7).

Let us remark that we will work with rows below. It is perhaps more natural to work with columns in the setting of vector bundles. Then the presented algorithm may be modified accordingly. In this case, one should start with constructing a matrix $\rho \in G_y$.

2.2 y-special matrices

For any $\sigma \in G_{xy}$, we expect to be able to construct matrices $\lambda \in G_x$ и $\rho \in G_y$ such that

$$\lambda\sigma\rho \in B_{xy}.$$

Let $\sigma \in G_{xy}$ be a given matrix, and let $\lambda \in G_{xy}$ be a result of its x-reduction.

Thus it will be important for our purposes to determine, whether the following condition

$$\lambda\sigma \in B_{xy}G_y \tag{9}$$

holds.

Let us introduce the class of y-special matrices.

2.2.1 Definition. *Let α be an xy-row. Then α is called y-**special** if the following conditions are satisfied:*

(i) *α is xy-unimodular.*

(ii) *α/y^* is y-unimodular (see (3)).*

*A matrix $\omega \in G_{xy}$ is said to be y-**special** if $\omega_{2,*}$ is y-special.*

By definition, any y-special matrix satisfies (9).

2.2.2. Speciality verification. Note that it can be very complicated to verify that a given row vector α over $A[x,y]$ is xy-unimodular. In our case, the latter condition holds for every α such that α is a row of a matrix in G_{xy}. Indeed, it follows from the completability of the row α to a matrix in G_{xy}. Thus we have only to determine whether an xy-unimodular row is y-special. But the latter is equivalent to the following condition:

$$\alpha/y^* \pmod{y} \quad \text{is a unimodular row over } A. \tag{10}$$

The last assertion follows from the observation that the open set U_1 and the zero locus of $t_1 \in \Gamma(\mathbf{P}^1_A, \mathcal{O}(1))$ form a cover of $\mathbf{P}^1_A$ (see §1).

2.2.3 Examples. Using the condition (10) we see that the matrix

$$\omega = \begin{bmatrix} 2y & 5 \\ 1 & 2x \end{bmatrix}$$

is not y-special.

In the case

$$\omega = \begin{bmatrix} 2x & 5 \\ 1 & 2y \end{bmatrix},$$

we obtain $\omega_{2,*}/y^* \pmod{y} = \begin{bmatrix} 1, & 0 \end{bmatrix}$. Hence, the y-speciality holds for ω.

Let us consider the matrix

$$\omega = \begin{bmatrix} -2x^3+1 & 4x^2 \\ -x^4 & 2x^3+1 \end{bmatrix}.$$

Since $\omega_{2,*}/y^* = \begin{bmatrix} -1, & 2y+y^4 \end{bmatrix}$ is unimodular over $\mathbb{Z}[y]$, it follows that ω is y-special.

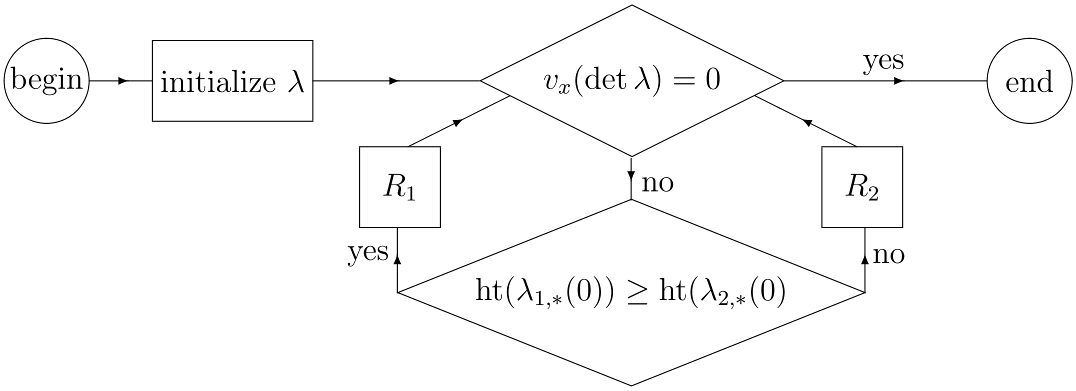
2.3 x-reduction

We next wish to find $\lambda \in G_x$ such that the product $\lambda\sigma$, where $\sigma \in G_{xy}$, is y-special.

2.3.1. Description. We sketch the main steps of the x-reduction procedure. We start by forming an initial λ. Note that if λ is in G_x then the x-reduction is finished. Otherwise we have to modify λ. To shorten notation we denote by R_1 and R_2 modifications of the first and second row of λ (see 2.3.3 and 2.3.4), respectively.

The remainder of the x-reduction can be done by performing R_1 and R_2 subroutines until the modified λ is invertible over $A[x]$. Let us remark that we are not required to store old λ's. By abuse of notation, the modified λ will again be denoted by λ.

Thus the x-reduction routine can be represented with the following diagram.



The initial λ is a matrix characterised by these properties:

$$\lambda \in G_{xy}; \tag{11}$$

$$\lambda \in M_x; \tag{12}$$

$$\lambda_{1,*}(0) \neq [0,0]; \quad \lambda_{2,*}(0) \neq [0,0]; \tag{13}$$

$$\lambda\sigma \text{ is } y\text{-special}. \tag{14}$$

Where by $\lambda_{i,*}(0)$ we denoted the evaluation of $\lambda_{i,*}$ at the point $x = 0$.

The x-reduction ends after a finite number of steps. In order to check this, we have to control the functions $v_x(\det \lambda)$, $\mathrm{ht}(\lambda_{1,*}(0))$, and $\mathrm{ht}(\lambda_{2,*}(0))$.

In fact, later we shall see that

$$R_2 \text{ strictly decreases } \mathrm{ht}(\lambda_{2,*}(0)); \tag{15}$$

$$R_1 \text{ leaves } \mathrm{ht}(\lambda_{2,*}(0)) \text{ unchanged}; \tag{16}$$

$$R_1 \text{ does not increase } v_x(\det \lambda); \tag{17}$$

$$R_1 \text{ decreases either } \mathrm{ht}(\lambda_{1,*}(0)), \text{ or } v_x(\det \lambda). \tag{18}$$

Also note that R_1 leaves $\lambda_{2,*}$ unchanged, and it may happen that R_1 increases $\mathrm{ht}(\lambda_{1,*}(0))$. Furthermore, R_2 does not change the first row of the matrix λ, and increases $v_x(\det \lambda)$.

It follows from the properties (15) and (16) that R_2 can not be executed more than h_2 times, where $h_2 = \mathrm{ht}(\lambda_{2,*}(0))$. Furthermore, R_1 can be executed only a finite number of times. Indeed, this follows from the properties (17) and (18). Consequently, the x-reduction process can always be completed in a finite number of steps.

The reason, why the x-reduction succeeds (see 2.3) goes as follows. Roughly, if the matrix λ becomes invertible over $A[x]$ then the x-reduction process is done. Otherwise, the rows of $\lambda(0)$ are linearly dependent. Consequently, we can apply the Euclidean algorithm.

More precisely, later we shall see that

$$R_1 \text{ preserves the properties (11), (12), and (14);} \tag{19}$$

$$R_2 \text{ preserves the properties (11), (12), and (14).} \tag{20}$$

We have seen (see the x-reduction flowchart) that the x-reduction is done whenever the condition $v_x(\det\lambda) = 0$ holds. From the latter condition, (11), and (12) it follows that $\lambda \in G_x$.

Since the property (14) holds for λ, we see that the x-reduction process succeeds.

2.3.2. Initialization of λ. The matrix σ^*/x^*, where σ^* is the adjugate to σ (see 1.3.2), satisfies (11), (12), (13), and (14). Moreover, it is evident that $\sigma^*/x^*\sigma \in T_{xy}$. Then we set

$$\lambda := \sigma^*/x^*.$$

Note that the choice of an initial λ is not unique.

2.3.3. R_1: modification of the first row of λ. The R_1 process modifies the first row of λ, whereas the second row of λ remains unchanged. Also, the input of R_1 consists of a matrix λ such that the conditions (11), (12), and

$$v_x(\det\lambda) \neq 0 \tag{21}$$

hold.

It follows from (12) that the row $\lambda(0)$ is well-defined. Besides, since λ satisfies (21), we see that $\lambda(0) = 0$, i.e. the rows $\lambda_{1,*}$ and $\lambda_{2,*}$ are linearly dependent. Hence, we can compute $[\lambda_{1,*}/\lambda_{2,*}]$ (see 1.4.4).

R_1 replaces $\lambda_{1,*}$ with the row $[\lambda_{1,*}/\lambda_{2,*}]$.

The final step of R_1 is the x-normalization of the modified matrix. More accurately, if $\lambda_{1,*}(0) = [0,0]$ then we have to apply the x-normalization. As mentioned before, it might happen that the height of the first row of the modified is increased. But in the latter case, $v_x(\det\lambda)$ decreases.

More explicitly, we describe steps of R_1 by the following formulas:

$$\lambda_{1,*} := \lambda_{1,*} - [\lambda_{1,*}(0)/\lambda_{2,*}(0)]\lambda_{2,*}; \tag{22}$$

$$\lambda_{1,*} := \lambda_{1,*}/x^* \text{ (see (1.3.3)).} \tag{23}$$

It is easily shown that R_1 has the properties (16), (17), (18), and (19) considered before. This is an immediate consequence of the above description. For instance, since R_1 does not change $\lambda_{2,*}$, we see that the second row of the product $\lambda\sigma$ remains unchanged.

Finally, R_1 is done as soon as the following inequality

$$\mathrm{ht}(\lambda_{1,*}) < \mathrm{ht}(\lambda_{2,*}) \tag{24}$$

is satisfied. Thus, we have to modify the second row of λ.

2.3.4. R_2: modification of the second row of λ. We shall decrease the height of the second row of λ in order to make $\lambda_{2,*}(0)$ unimodular. The main problem we encounter is the fact that a modification of the second row of λ changes the second row of the product $\lambda\sigma$. In other words, we need to ensure that the modified λ is a y-special matrix.

We change the limiting behaviour of $\lambda_{2,*}$ as $x \to 0$, whereas the behaviour of $\lambda_{2,*}$ will remain unchanged. Precisely the latter behaviour affects the y-speciality.

We describe steps of R_2 by the following formulas:

$$n := \max\{1, v_y(\lambda_{2,*}\sigma) - v_y(\lambda_{1,*}\sigma) + 1\}; \tag{25}$$

$$\lambda_{2,*} := x^n\lambda_{2,*} + \lambda_{1,*}. \tag{26}$$

Let us remark that in (26) one may choose any positive $n \in \mathbb{Z}$ sufficiently large.

The R_2 subroutine works with the matrix λ and $v_y(\lambda_{2,*}\sigma)$, $v_y(\lambda_{1,*}\sigma) \in \mathbb{Z}$. In this case, it is more natural to consider that R_2 uses λ and σ, whereas R_2 modifies only the second row of λ. Note that the matrix σ remains unchanged.

R_2 clearly has the properties (15) and (20). This is an immediate consequence of the R_2 subroutine description. For instance, (15) follows from (24) and the positivity of n. Since $n > v_y(\lambda_{2,*}\sigma) - v_y(\lambda_{1,*}\sigma)$, we have that the product $\lambda\sigma$ satisfies (14) after the modification of λ. The latter statement is a part of (20).

2.4 y-reduction

It should be pointed out that although the Euclidean condition is not necessary to the following procedure, the Euclidean algorithm may be helpful. Moreover, it is sufficient to require A to be a principal ideal domain.

As already noted in 2.1, the y-reduction routine takes as input $\sigma \in G_{xy}$ and $\lambda \in G_x$ such that the y-speciality condition (see 2.2.1) holds for $\lambda\sigma$. The resulting output $\rho \in G_y$ satisfies $\lambda\sigma\rho \in B_{xy}$, i.e.

$$\lambda\sigma\rho = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix},$$

where by $*$ we denoted arbitrary elements of $A[x, y]$.

2.4.1. y-reduction description. Consider the y-normalization of $\lambda\sigma$: namely, set

$$\eta := (\lambda\sigma)/y^* \in M_y.$$

Recall that the y-normalization operation is introduced in 1.3.3. Thus we have

$$\lambda\sigma = \delta\eta, \qquad \text{where } \delta \in T_{xy}.$$

Since the matrix $\lambda\sigma$ is y-special, the second row of η is y-unimodular by definition. Assume now that $\eta_{2,*}$ is completed to a matrix $\theta \in G_y$. Let ρ be defined by

$$\rho = \theta^{-1}.$$

In this case, we have a chain of equalities

$$\lambda\sigma\rho = \delta\eta\theta^{-1} = \delta \begin{bmatrix} * & * \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix},$$

so that $\lambda\sigma\rho$ is upper triangular, as desired. This follows immediately from the fact that the second row of $\eta\theta^{-1}$ depends only on the second row of η. Thus the second row of $\eta\theta^{-1}$ coincides with the second row of the product $\theta\theta^{-1} = 1$.

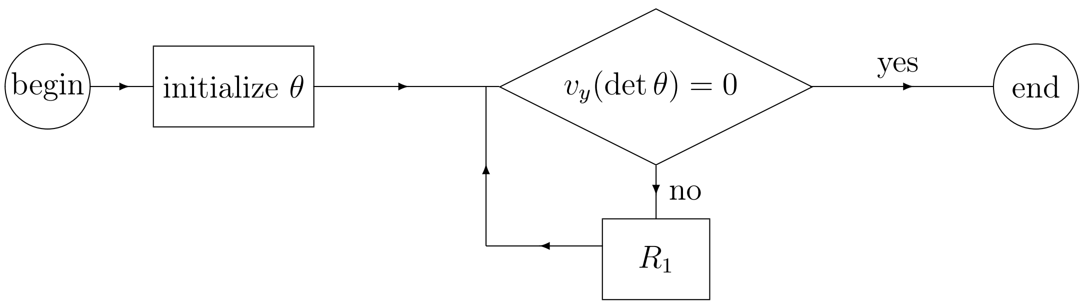
Consequently, to reduce σ to upper triangular form, it suffices to find such θ. For this reason, we introduce a y-reduction routine.

It remains to complete given y-unimodular row to an invertible matrix over $A[y]$. The construction uses one of the subroutines E and S which will be introduced in Sections 2.4.2 and 2.4.3 below. Note that it is far more convenient to deal with the matrix η instead of working only with $\eta_{2,*}$.

The E subroutine requires A to be a Euclidean domain. In fact, it is based on the R_1 subroutine introduced in 2.3.3.

On the other hand, given any A-unimodular row, it can be completed to an invertible matrix over A. The latter observation gives rise to the S subroutine.

2.4.2. E: completing $\eta_{2,*}$ to $\theta \in G_y$. A is assumed here to be a Euclidean domain. Let η be a matrix over $A[x]$ such that $\eta \in G_{xy}$, $\eta_{1,*}(0) \neq [0,0]$, and $\eta_{2,*}$ is y-unimodular. The E subroutine takes such η as input. The structure of E can be represented with the following diagram:



θ can be initialized as follows

$$\theta = \eta.$$

The description of R_1 in 2.3.3 can easily be modified: namely, we replace λ by η and x by y. In this case, by $\eta(0)$ we denoted the evaluation of η at the point $y = 0$.

Moreover, E can be executed only a finite number of times by the arguments given above (see the end of 2.3.3). It is readily verified that the resulting matrix θ has the desired properties.

2.4.3. S: completing a y-row $\eta_{2,*}$ to $\theta \in G_y$. Let A be a PID in which every unimodular row is completable. Let η be a matrix over $A[x]$ such that $\eta \in G_{xy}$, $\eta_{1,*}(0) \neq [0, 0]$, and $\eta_{2,*}$ is y-unimodular.

We construct θ as follows

$$\theta = \alpha\gamma, \text{ where } \gamma \in G(A). \tag{27}$$

First note that γ can be obtained by completing the unimodular row $\eta_{2,*}(0)$ to an invertible matrix over A.

Further, define the initial α by

$$\alpha = \eta\gamma^{-1}. \tag{28}$$

Note that

$$\alpha(0) = \begin{bmatrix} * & * \\ 0 & 1 \end{bmatrix},$$

where by $*$ we denoted arbitrary elements of A. We also note that γ was introduced for reducing $\alpha(0)$ to such form. It remains to modify α to obtain an invertible matrix. We introduce the following steps:

(i) Let $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(ii) Then set

$$\alpha := \alpha_1/y^*, \text{ where } \alpha_1 = \begin{bmatrix} a - b(0)c & b - b(0)d \\ c & d \end{bmatrix}.$$

More precisely, we have to repeat the process until α is not invertible. In other words, one has to check whether $v_y(\det\alpha) = 0$ before entering the loop body. Finally, to find η, we use the resulting α and (27).

The same arguments as in 2.3.3 can be employed to prove the finiteness of the described process.

2.5 Examples of constructed filtrations

Let $A = \mathbb{Z}$ and $\mathrm{ht}(n) = |n|$ (see 1.4 for the definition). We shall now consider three examples. In order to illustrate a work of the algorithm, Example 2.5.1 is analyzed in detail. In the second example we demonstrate a slight modification of the algorithm (according to the remark in 2.3.4). Finally, we sketch more complicated Example 2.5.3.

2.5.1. Let E be a vector bundle defined by the gluing matrix

$$\sigma = \begin{bmatrix} 2x^{-1} & 5 \\ 1 & 2x \end{bmatrix}.$$

We now apply the algorithm to σ (see 2). We start by performing the x-reduction to construct λ. In this case, the initial λ is defined by (see the flowchart 2.3.1 and 2.3.2):

$$\lambda = \sigma^*/x^* = \begin{bmatrix} 2x & -5 \\ -1 & 2x^{-1} \end{bmatrix} /x^* = \begin{bmatrix} 2x & -5 \\ -x & 2 \end{bmatrix}. \tag{29}$$

Here we have $\det \lambda = -x$ and $v_x(\det \lambda) = 1$. Hence, the condition $v_x(\det \lambda) = 0$ is not satisfied.

Therefore we have to check whether the following inequality holds

$$\mathrm{ht}(\lambda_{1,*}(0)) \geq \mathrm{ht}(\lambda_{2,*}(0)) \text{ (see the flowchart in 2.3.1).}$$

Evaluating the rows of λ at the point $x = 0$, we see that the latter condition is satisfied (namely, $5 \geq 2$). Hence, we have to apply the R_1 subroutine (see 2.3.3).

Further, choose generators $u = -5$, $v = 2$ of the ideals $(0, -5)$ and $(0, 2)$, respectively. They agree in the sense of (5). Now choose $[u/v]$ to be equal -2.

Thus, we have

$$[\lambda_{1,*}(0)/\lambda_{2,*}(0)] = [[0, -5]/[0, 2]] = -2.$$

By applying the formulas (22) and (23) to λ (see (29)), we get a new λ, namely this can be done as follows:

$$R_1 : \begin{bmatrix} 2x & -5 \\ -x & 2 \end{bmatrix} \overset{md}{\rightsquigarrow} \begin{bmatrix} 0 & -1 \\ -x & 2 \end{bmatrix} /x^* = \begin{bmatrix} 0 & -1 \\ -x & 2 \end{bmatrix}, \tag{30}$$

where by *md* we mean the modification of λ by adding the second row multiplied by 2 to the first row of λ.

We now continue the preceding construction by verifying the conditions in the diagram depicted in 2.3.1. The verification shows that we have to perform the R_2 subroutine (see 2.3.4). As a first step, we compute

$$\lambda\sigma = \begin{bmatrix} 0 & -1 \\ -x & 2 \end{bmatrix} \begin{bmatrix} 2x^{-1} & 5 \\ 1 & 2x \end{bmatrix} = \begin{bmatrix} -1 & -2x \\ 0 & -x \end{bmatrix}.$$

In this case, we have

$$\max\{1, v_y(\lambda_{2,*}\sigma) - v_y(\lambda_{1,*}\sigma) + 1\} = \max\{1, (-1) - (0) + 1\} = 1,$$

and we set

$$n = 1. \tag{31}$$

We obtain a new λ by applying the formula (26) to the matrix λ computed in (30). More explicitly, a modified λ is constructed as follows:

$$R_2 : \begin{bmatrix} 0 & -1 \\ -x & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & -1 \\ -x^2 & -1 + 2x \end{bmatrix},$$

i.e. the second row multiplied by $x^n = x$ is added to the first row.

The remainder of the x-reduction routine will be sketched by the following chain of modifications:

$$\begin{bmatrix} 0 & -1 \\ -x^2 & -1+2x \end{bmatrix} \overset{R_1}{\rightsquigarrow} \begin{bmatrix} x & -2 \\ -x^2 & -1+2x \end{bmatrix} \overset{R_1}{\rightsquigarrow} \begin{bmatrix} 1+2x & -4 \\ -x^2 & -1+2x \end{bmatrix}. \tag{32}$$

We see that $\det\lambda = -1$ and $v_x(\det\lambda) = 0$, where λ is given by the right hand side of (32). Consequently, the desired λ is constructed.

Our next goal is to find ρ (see the beginning of 2.1). To begin the y-reduction (see 2.4), we set

$$\eta = (\lambda\sigma)/y^* = \begin{bmatrix} 2y^2 & 5y+2 \\ -y^2 & -2y-1 \end{bmatrix}.$$

Recall that we have to complete the row $\eta_{2,*} = [-y^2, -2y-1]$ to an invertible matrix θ over $A[y]$. Since $\mathbb{Z}$ is Euclidean, it follows that both subroutines introduced in 2.4 are applicable in this case.

We first find λ using the S subroutine (see the description in 2.4.3). We have $\eta(0)_{2,*}(0) = [0, -1]$. Let

$$\gamma = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We start with the initialization of α

$$\alpha = \eta\gamma^{-1} = \begin{bmatrix} -2y^2 & -5y-2 \\ y^2 & 2y+1 \end{bmatrix}.$$

The remainder of the S subroutine may be depicted as follows.

$$\alpha = \begin{bmatrix} -2y^2 & -5y-2 \\ y^2 & 2y+1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & -1 \\ y^2 & 2y+1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} y & 2 \\ y^2 & 2y+1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1-2y & -4 \\ y^2 & 2y+1 \end{bmatrix}.$$

Since the latter matrix satisfies $\det\alpha = 1$, it follows that S is done. Then we have

$$\theta = \alpha\gamma = \begin{bmatrix} -1+2y & 4 \\ -y^2 & -2y-1 \end{bmatrix}, \text{ and } \rho = \theta^{-1} = \begin{bmatrix} -1-2y & -4 \\ y^2 & -1+2y \end{bmatrix}.$$

Thus the matrices $\lambda \in G_x$ and $\rho \in G_y$ are constructed, so that we have a decomposition

$$\sigma' = \lambda\sigma\rho = \begin{bmatrix} 2x^{-1} & 5+2x \\ -1 & -2x-x^2 \end{bmatrix} \begin{bmatrix} -1-2y & -4 \\ y^2 & -1+2y \end{bmatrix} = \begin{bmatrix} x^{-2} & -1+2x^{-1} \\ 0 & x^2 \end{bmatrix}.$$

In the setting of vector bundles, E fits into the following exact sequence

$$0 \to \mathcal{O}(-2) \to E \to \mathcal{O}(2) \to 0.$$

2.5.2. We now consider a bundle E as above, i.e. E is defined by the matrix σ from 2.5.1. We shall repeat several steps of 2.5.1, but choose a different n in (31) to illustrate the remark in the description of R_2. Set

$$n = 2. \tag{33}$$

Then the desired λ can be obtained from the following chain of modifications:

$$\begin{bmatrix} 0 & -1 \\ -x & 2 \end{bmatrix} \overset{R_2}{\rightsquigarrow} \begin{bmatrix} 0 & -1 \\ -x^3 & -1+2x^2 \end{bmatrix} \overset{R_1}{\rightsquigarrow} \begin{bmatrix} x & -2 \\ -x^3 & -1+2x^2 \end{bmatrix} \overset{R_1}{\rightsquigarrow} \begin{bmatrix} 1+2x^2 & -4x \\ -x^3 & -1+2x^2 \end{bmatrix}.$$

We start the y-reduction by defining

$$\eta = (\lambda\sigma)/y^* = \begin{bmatrix} 2y^3 & 5y^2+2 \\ -y^3 & -2y^2-1 \end{bmatrix}.$$

For example, we take $\gamma = -1$. Further, α is defined by the formula $\alpha = \eta\gamma^{-1}$.

We now sketch the remainder of the y-reduction.

$$\begin{bmatrix} -2y^3 & -5y^2-2 \\ y^3 & 2y^2+1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & -1 \\ y^3 & 2y^2+1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} y & 2 \\ y^3 & 2y^2+1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1-2y^2 & -4y \\ y^3 & 2y^2+1 \end{bmatrix}.$$

Since the latter matrix is invertible over $A[y]$, it follows that the desired α is constructed. Then we have

$$\theta = \alpha\gamma = \begin{bmatrix} -1+2y^2 & 4y \\ -y^3 & -2y^2-1 \end{bmatrix} \text{ and } \rho = \theta^{-1} = \begin{bmatrix} -1-2y^2 & -4y \\ y^3 & -1+2y^2 \end{bmatrix}.$$

Finally, we obtain a decomposition

$$\sigma' = \lambda\sigma\rho = \begin{bmatrix} x^{-3} & 2x^{-2}+1-2x^2 \\ 0 & x^3 \end{bmatrix}.$$

Thus E fits into the exact sequence

$$0 \to \mathcal{O}(-3) \to E \to \mathcal{O}(3) \to 0.$$

2.5.3. Let E be a vector bundle defined by the following matrix

$$\sigma = \begin{bmatrix} 5 & 17 \\ 2x & 7x \end{bmatrix}.$$

We next define $\lambda = \sigma^*/x^*$. Further computations are sketched as follows.

$$\begin{bmatrix} 7x & -17 \\ -2x & 5 \end{bmatrix} \overset{R_1}{\rightsquigarrow} \begin{bmatrix} x & -2 \\ -2x & 5 \end{bmatrix} \overset{R_2}{\rightsquigarrow} \begin{bmatrix} x & -2 \\ x-2x^2 & -2+5x \end{bmatrix} \overset{R_1}{\rightsquigarrow} \begin{bmatrix} 2x & -5 \\ x-2x^2 & -2+5x \end{bmatrix} \overset{R_1}{\rightsquigarrow}$$

$$\overset{R_1}{\rightsquigarrow} \begin{bmatrix} 4x^2 & -1-10x \\ x-2x^2 & -2+5x \end{bmatrix} \overset{R_2}{\rightsquigarrow} \begin{bmatrix} 4x^2 & -1-10x \\ 4x^2+x^3-2x^4 & -1-10x-2x^2+5x^3 \end{bmatrix} \overset{R_1}{\rightsquigarrow}$$

$$\begin{bmatrix} -x+2x^2 & 2-5x \\ 4x^2+x^3-2x^4 & -1-10x-2x^2+5x^3 \end{bmatrix} \overset{R_1}{\rightsquigarrow}$$

$$\overset{R_1}{\rightsquigarrow} \begin{bmatrix} -1+10x+2x^2-4x^3 & -25-4x+10x^2 \\ 4x^2+x^3-2x^4 & -1-10x-2x^2+5x^3 \end{bmatrix}.$$

We see that $\det(\lambda) = 1$. Consequently, λ is constructed. We next compute

$$\lambda\sigma = \begin{bmatrix} -5+2x^2 & -17-5x+6x^2+2x^3 \\ -2x+x^3 & -7x-2x^2+3x^3+x^4 \end{bmatrix},$$

and define

$$\eta = (\lambda\sigma)/y^* = \begin{bmatrix} -5y^3+2y & -17y^3-5y^2+6y+2 \\ -2y^3+y & -7y^3-2y^2+3y+1 \end{bmatrix}.$$

Taking $\theta = \eta$ and applying the E subroutine, we obtain a chain of modifications

$$\begin{bmatrix} -5y^3+2y & -17y^3-5y^2+6y+2 \\ -2y^3+y & -7y^3-2y^2+3y+1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -y & -3y-1 \\ -2y^3+y & -7y^3-2y^2+3y+1 \end{bmatrix} \rightsquigarrow$$

$$\begin{bmatrix} -2y & -7y-2 \\ -2y^3+y & -7y^3-2y^2+3y+1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -4y^2 & -14y^2-4y-1 \\ -2y^3+y & -7y^3-2y^2+3y+1 \end{bmatrix} \rightsquigarrow$$

$$\begin{bmatrix} -2y^2-4y+1 & -7y^2-16y-1 \\ -2y^3+y & -7y^3-2y^2+3y+1 \end{bmatrix}.$$

It easy to see that the latter matrix is invertible over $A[y]$; thus the required λ is constructed. It remains to define ρ by

$$\rho = \theta^{-1} = \begin{bmatrix} -7y^3-2y^2+3y+1 & 7y^2+16y+1 \\ 2y^3-y & -2y^2-4y+1 \end{bmatrix}.$$

An explicit computation shows that E fits into the following exact sequence

$$0 \to \mathcal{O}(-3) \to E \to \mathcal{O}(4) \to 0.$$

3 Conclusion

It is natural to ask if there exists an algorithm for finding minimal filtrations, or, equivivalently, given a vector bundle E of rank 2, we wish to find a subbundle $L \subset E$ such that the number $2\deg L - \deg E$ is maximal. A question arises: whether this can be done using the introduced algorithm or its slightly modified forms? For example, an initial λ in 2.3.2 may be chosen in a more careful way.

It is proved in [12] that any vector bundle on $\mathbf{P}^1_{\mathbb{Z}}$ with the generic fiber $\mathcal{O}^2$ and simple jumps has $\mathcal{O}(-2)$ as a subbundle. Example 2.5.1 deals with a bundle E of such a form. It can be also obtained that $\mathcal{O}(-1) \not\subseteq E$. Thus the constructed filtration is minimal in the sense defined above.

References

[1] *Ch. C. Hanna.* Subbundles of vector bundles on the projective line. J. Algebra, 52, no. 2, 322-327, 1978.

[2] *A. N. Parshin.* Higher Bruhat–Tits buildings and vector bundles on an algebraic surface, Göttingen, Preprint, 1993.

[3] *A. L. Smirnov.* On filtrations of vector bundles over $\mathbf{P}^1_{\mathbb{Z}}$, Arithmetic and Geometry, Cambridge Univ. Press, London Math. Soc. Lect. Note Series, vol. 420 (2015), 436-457.

[4] *J.-P. Serre.* Trees, Springer-Verlag, 1980.

[5] *R. Hartshorne.* Algebraic Geometry, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, New York (1977)

[6] *C. Okonek, M. Schneider, H. Spindler.* Vector Bundles on Complex Projective Spaces, Progress in Mathematics, vol. 3, Birkhäuser, Boston, MA, 1980.

[7] *J.-P. Serre.* Faiseaux Algébriques Cohérents, The Ann. of Math., 2nd Ser., Vol. 61, No. 2. (Mar., 1955), pp. 197-278.

[8] *G. Horrocks.* Projective modules over an extension of a local ring, Proc. London Math. Soc. (3) 14 (1964), 714-718.

[9] *B. L. van der Waerden.* Algebra. Vol 1, Translated by Fred Blum and John R. Schulenberger, Springer-Verlag New York, 1991

[10] *D. Quillen.* Projective modules over polynomial rings, Invent. Math., 1976, vol. 36, p. 167–171.

[11] *A. Suslin.* Projective modules over polynomial rings are free, Doklady Akademii Nauk SSSR (in Russian), 1976, 229 (5), p. 1063–1066.

[12] *A. L. Smirnov.* Vector $\mathbf{P}^1_{\mathbb{Z}}$-bundles with simple jumps, Preprints PDMI, 01/2015.