Fifth exercise sheet "Algebra II" winter term 2024/5.

Problem 1 (2 points)**.** *Let R be a DVR and S its integral closure in a finite purely inseparable extension L of its field of quotients K. Show that S is a DVR.*

Problem 2 (3 points)**.** *Let R be a ring such that for every $r \in R \setminus \{0\}$, there are only finitely many $\mathfrak{p} \in \operatorname{Spec} R$ containing r, and such that all $R_\mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Spec} R \setminus \big\{\{0\}\big\}$ are Noetherian. Show that R is Noetherian.*

Problem 3 (3 points)**.** *Let R be a Dedekind domain and S its integral closure in a finite purely inseparable field extension of its field of quotients K. Show that S is a Dedekind domain.*

Remark 1. *Together with the results shown in the lecture this implies that the integral closure if R in any finite field extension of K is a Dedekind domain. This is sufficient for all applications of Theorem 6 in the lecture.*

Problem 4 (5 points)**.** *Let R be a Dedekind domain which is not a field. Let $I_{1,2} = \prod_{\mathfrak{m} \in \operatorname{mSpec} R} \mathfrak{m}^{d_{\mathfrak{m},i}}$ be two fractional ideals of R. Show that $I_1 \subseteq I_2$ if and only if $d_{\mathfrak{m},1} \geq d_{\mathfrak{m},2}$ for all $\mathfrak{m} \in \operatorname{mSpec} R$.*

Problem 5 (7 points)**.** *Let $K = \mathbb{Q}\big(\sqrt{D}\big)$ where $D \neq 1$ is a square-free integer. Calculate the prime ideal decomposition of $2\mathcal{O}_K$.*

Remark 2. *Of course this continues Problem 2 of the previous sheet. As a correct solution to this problem is therefore a prerequisite to the solution of the current Problem 5, it must be discussed in the exercise sessions this week. If a correct base of $\mathcal{O}_K$ is known the finite ring $R = \mathcal{O}_K/2\mathcal{O}_K$ can be determined, and this carries all the needed information about the desired prime ideal decomposition which by Remark 2.5.3 from the lecture has the form*

$$2\mathcal{O}_K = \prod_{i=1}^{N} \mathfrak{m}_i^e.$$

If $N > 1$ then by the Chinese Remainder Theorem there are non-trivial ($\varepsilon \notin \{0; 1\}$) idempotents (i. e, $\varepsilon^2 = \varepsilon$) $\varepsilon \in R$. If $e > 1$ there are nilpotent elements.

Problem 6 (2 points)**.** *Use the result of the previous problem to show that there are infinitely many number fields with even class number.*

Remark 3. *The finiteness of the class number will be shown later on. For the purposes of this exercise sheet, "even class number" thus translates into the existence of an element of order 2 in the ideal class group of $\mathcal{O}_K$.*

Two of the 22 points for this sheet are bonus points which do not count in the calculation of the 50%-bound for passing the exercises. Solutions should be submitted to the tutor by e-mail before Friday November 15 24:00.