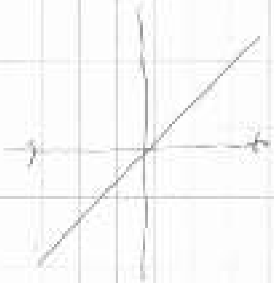
§5: (Einige) Anwendungen

① a) $f(x) = x$ auf $[-\pi, \pi]$



$$f(x) = \sum_{\nu=1}^{\infty} a_\nu \sin \nu x$$

$$a_\nu = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin \nu x \, dx$$

$$= \frac{1}{\pi} \Big[-x \frac{\cos \nu x}{\nu} \Big|_{-\pi}^{\pi} + \underbrace{\frac{1}{\nu} \int_{-\pi}^{\pi} \cos \nu x \, dx}_{=0} \Big]$$

$$= 2 \cdot \frac{(-1)^{\nu+1}}{\nu}$$

$$\leadsto x = 2 \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu+1}}{\nu} \sin \nu x$$

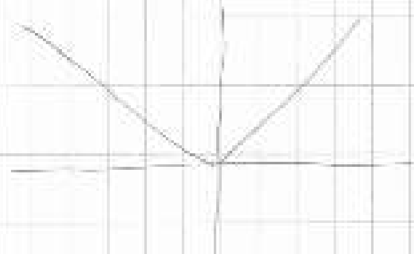
lokal glm. auf $(-\pi, \pi)$

Setze $x = \frac{\pi}{2}$

Satz 1 $\quad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \pm \ldots$

(Leibniz, „numero deus impari gaudet"

b) $f(x) = |x|$



$$f(x) = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} a_\nu \cos \nu x$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi$$

$$a_\nu = \frac{2}{\pi}\int_0^{\pi} x \cos\nu x\,dx$$

$$= \frac{2}{\pi}\Big[\underbrace{\frac{1}{\nu}\, x \sin\nu x\Big|_0^{\pi}}_{=0} - \frac{1}{\nu}\int_0^{\pi} \sin\nu x\,dx\Big]$$

$$= \frac{2}{\pi}\frac{1}{\nu^2}\cos\nu x\Big|_0^{\pi} = \frac{2}{\pi\nu^2}\left(-1+(-1)^\nu\right)$$

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{\nu=1}^{\infty}\frac{1}{(2\nu+1)^2}\cos(2\nu+1)x$$

Folgerung: Setze $x = 0$

Satz 2 $\quad \dfrac{\pi^2}{8} = 1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$ (Euler)

Umformungen:

$$\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)\left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \ldots\right)$$

$$= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \ldots$$

$$\frac{\pi^2}{8}\,\frac{1}{1-1/4} = \sum_{\nu=1}^{\infty}\frac{1}{\nu^2}$$

Satz 3 (Euler) $\quad \dfrac{\pi^2}{6} = \displaystyle\sum_{\nu=1}^{\infty}\frac{1}{\nu^2}$