Mirror symmetry and stability conditions on K3 surfaces

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1. Gutachter: Prof. Dr. Daniel Huybrechts
2. Gutachter: Prof. Dr. Emanuele Macrì

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Summary

This thesis is concerned with questions arising in the realm of mirror symmetry for K3 surfaces. It is divided into two parts.

In the first part, we study the bijection between Fourier–Mukai partners of a K3 surface X and cusps of the Kähler moduli space which was established by Shouhei Ma in [Ma09]. The Kähler moduli space can be described as a quotient of Bridgeland's manifold of stability conditions on the derived category of coherent sheaves $\mathcal{D}^b(X)$. We relate stability conditions σ near to a cusp and the associated Fourier–Mukai partner Y in the following ways.

1. We construct a special path of stability conditions $\sigma(t)$ such that the hearts converge to the heart $Coh(Y)$ of coherent sheaves on Y.

2. We study a class of geodesic degenerations of stability conditions $\sigma(t)$ and show that the hearts of $\sigma(t)$ are related to $Coh(Y)$ by a tilt.

3. We construct Y as moduli space of σ-stable objects.

On the way of proving the these results, we establish some properties of the group of auto-equivalences of $\mathcal{D}^b(X)$ which respect the component $Stab^\dagger(X)$ of the stability manifold.

In the second part, we provide an explicit example of mirror symmetry for K3 surfaces in the sense of Aspinwall and Morrison [AM97]. Consider the quartic K3 surface in $\mathbb{P}^3$ and its mirror family obtained by the orbifold construction.

1. We give an explicit computation of the Hodge structures and period maps for these families of K3 surfaces.

2. We identify a mirror map, i.e. an isomorphism between the complex and symplectic deformation parameters, and explicit isomorphisms between the Hodge structures, which classify the conformal field theories.

Our results rely on earlier work by Narumiyah–Shiga [NS01], Dolgachev [Dol96] and Nagura–Sugiyama [NS95].

Contents

To my parents

Chapter 1

Introduction

In this thesis we study questions which are related to mirror symmetry for K3 surfaces. Before we review the mirror conjectures for K3 surfaces and explain our own results we comment briefly on the more familiar picture of mirror symmetry for 3-dimensional Calabi–Yau manifolds.

1.1 Mirror symmetry for Calabi–Yau 3-folds

Mirror symmetry is a duality between certain super-symmetric conformal field theories which are associated to Calabi–Yau manifolds via string theory. More precisely, a Calabi–Yau manifold X with chosen Kähler form ω_X determines two topological conformal field theories $CFT_A(X)$ and $CFT_B(X)$. The essential fact is, that $CFT_A(X)$ only depends on the symplectic structure induced by ω_X and $CFT_B(X)$ only depends on the complex structure of X. Two Calabi–Yau manifolds X, Y are called *mirror pair* if there is an isomorphism

$$CFT_B(X) \cong CFT_A(Y). \tag{MS}$$

Although the above process of associating a conformal field theory to a Calabi–Yau manifold is not rigorously established, there are several interesting conjectures which are motivated by the physical arguments.

The most famous example is the homological mirror symmetry conjecture by M. Kontsevich [Kon95], which states that for a mirror pair of Calabi–Yau manifolds X, Y we have an equivalence of triangulated (A_∞-) categories

$$\mathcal{D}^b(X) \cong \mathcal{D}^\pi Fuk(Y). \tag{HMS}$$

Here $\mathcal{D}^b(X)$ is the derived category of coherent sheaves and $\mathcal{D}^\pi Fuk(Y)$ is the derived Karoubian-closed Fukaya category of Y. Recall that the Fukaya category is an invariant of the symplectic structure ω_Y of Y and $\mathcal{D}^b(X)$ depends only on the complex structure of X.

Furthermore, it is expected that for a mirror pair X, Y of Calabi–Yau 3-folds there is an isomorphism, called mirror map, between the complex deformation space of X and the Kähler- or symplectic-deformation space of Y:

$$\mathrm{CDef}(X) \cong \mathrm{KDef}(Y). \tag{DEF}$$

More precisely, such an isomorphism is expected only near maximal degeneration points in the moduli space of complex structures and, moreover, one has to consider symplectic structures which are *complexified* using a so called *B-field* $\beta \in H^2(Y, \mathbb{R})$.

The mirror map identifies certain functions on these deformation spaces. On the complex side there is the Yukawa coupling, which is defined in terms of Hodge theory. On the symplectic side there is the Gromov–Witten potential, which encodes information about rational curves of given degree on Y. This property was used in [COGP91] to make a spectacular prediction for the numbers of rational curves on a quintic threefold in $\mathbb{P}^4$.

1.2 Mirror symmetry for K3 surfaces

In the case of K3 surfaces there are some differences to the above picture. Firstly, Aspinwall and Morrison [AM97] give a classification of the conformal field theories associated to K3 surfaces in terms of Hodge structures (cf. chapter 8). To a K3 surface X with chosen Kähler form ω_X they associate two weight-two Hodge structures $H_A(X, \mathbb{Z})$ and $H_B(X, \mathbb{Z})$, such that (MS) is equivalent to the existence of a Hodge isometry

$$H_B(X, \mathbb{Z}) \cong H_A(Y, \mathbb{Z}). \tag{AM}$$

We will take this as definition of mirror symmetry. In the second part of this thesis we will construct explicitly such a Hodge isometry for the quartic surface in $\mathbb{P}^3$ and its mirror family.

The homological mirror symmetry conjecture (HMS) still makes sense and is expected to hold also for K3 surfaces.

The isomorphism (DEF) has to be modified. This is due to the fact that for a K3 surface $h^{2,0}(X) \neq 0$ and therefore complex and symplectic deformations do not "decouple" [AM97]. One can circumvent this problem by restricting the class of admitted deformations using a *lattice polarization* [Dol96] (cf. section 8.5). An alternative approach is to parametrize complex and symplectic structures at the same time using the concept of a *generalized K3 surface* due to Hitchin (cf. [Huy05], [Huy04]).

In contrast to the 3-dimensional situation, one expects that the mirror map extends to a global isomorphism of moduli spaces

$$CM(X) \cong KM(Y). \tag{MM}$$

Here $CM(X)$ is the moduli space of $NS(X)$-polarized K3 surfaces and $KM(Y)$ is the global complexified Kähler moduli space of Y. As explained below, the Kähler moduli space has a natural description as a quotient of Bridgeland's manifold of stability conditions on $\mathcal{D}^b(Y)$.

1.3 Cusps of the Kähler moduli space and stability conditions

The first part of this thesis is concerned with the study of the Kähler moduli space of a projective K3 surface Y over the complex numbers. Let $\mathcal{T} = \mathcal{D}^b(Y)$ be the bounded derived category of coherent sheaves on Y.

The complexified Kähler moduli space $KM(Y) = KM(\mathcal{T})$ is defined as follows. Let $N(\mathcal{T})$ be the numerical Grothendieck group of $\mathcal{T}$ endowed with the (negative) Euler pairing. We consider the following period domain

$$\mathfrak{D}(\mathcal{T}) = \{[z] \in \mathbb{P}(N(\mathcal{T})_{\mathbb{C}}) \,|\, z.z = 0, z.\bar{z} > 0\}.$$

and define $KM(\mathcal{T})$ to be a connected component of $Aut(\mathcal{T}) \setminus \mathfrak{D}(\mathcal{T})$. There is a canonical open embedding of the complexified Kähler cone into the Kähler period domain:

$$Amp(Y) \times NS(Y)_{\mathbb{R}} \hookrightarrow \mathfrak{D}(\mathcal{T}).$$

The image of $Aut(\mathcal{T})$ in the orthogonal group $O(N(\mathcal{T}))$ is known by [HMS09]. In particular it is an arithmetic subgroup. Therefore, we can compactify the Kähler moduli space to a projective variety $\overline{KM}(\mathcal{T})$ using the Baily–Borel construction [BB66]. The boundary $\overline{KM}(\mathcal{T}) \setminus KM(\mathcal{T})$ consists of components, called cusps, which are divided into the following types (cf. chapter 3):

- 0-dimensional standard cusps,
- 0-dimensional cusps of higher divisibility and
- 1-dimensional boundary components.

In [Ma09], [Ma10] Shouhei Ma establishes a bijection between

$$\left\{ \begin{array}{c} \text{K3 surfaces } Z \\ \text{with } \mathcal{D}^b(Z) \cong \mathcal{T} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{standard cusps of the} \\ \text{Kähler moduli space } KM(\mathcal{T}) \end{array} \right\}. \qquad (*)$$

Moreover, cusps of higher divisibility correspond to realizations of $\mathcal{T}$ as the derived category of sheaves on a K3 surface twisted by a Brauer class. Unfortunately the proof is not geometric but uses deep theorems due to Mukai and Orlov to translate the statement into lattice theory.

The aim of the first part of this thesis is to find a more geometric explanation for this phenomenon using the theory of stability conditions due to Bridgeland [Bri07], [Bri08]. The space $Stab(\mathcal{T})$ of Bridgeland stability conditions on $\mathcal{T}$ is a complex manifold and carries canonical actions of $Aut(\mathcal{T})$ and of the universal cover $\widetilde{Gl}_2^+(\mathbb{R})$ of $Gl_2^+(\mathbb{R})$. For each pair of $\omega \in Amp(Y)$ and $\beta \in NS(Y)_{\mathbb{R}}$ with $\omega^2 > 2$ Bridgeland constructs an explicit stability condition $\sigma_Y(\beta, \omega) \in Stab(\mathcal{T})$. Denote by $Stab^\dagger(\mathcal{T})$ the connected component of $Stab(\mathcal{T})$ containing these stability conditions.

A special open subset of $KM(\mathcal{T})$ can be identified with the quotient space

$$KM_0(\mathcal{T}) \cong Aut^\dagger(\mathcal{T}) \setminus Stab^\dagger(\mathcal{T}) / \widetilde{Gl}_2^+(\mathbb{R}), \qquad (1.1)$$

where $Aut^\dagger(\mathcal{T})$ is the group of auto-equivalences respecting the distinguished component $Stab^\dagger(\mathcal{T})$. This statement is essentially due to Bridgeland and was stated in [Ma09] and [Bri09] before. However, it seems to rely on properties of the group $Aut^\dagger(\mathcal{T})$ which are established in chapter 5, cf. Corollary 5.13. We denote the quotient map by $\pi : Stab^\dagger(\mathcal{T}) \to KM(\mathcal{T})$.

Our first result addresses the following question: Every stability condition σ determines a heart $\mathcal{A}(\sigma)$ of a bounded t-structure. Also, every derived equivalence $\Phi : \mathcal{D}^b(Z) \xrightarrow{\sim} \mathcal{T}$ determines the heart $\Phi(Coh(Z))$. How are these two hearts related for $\pi(\sigma)$ near the cusp associated to Z?

Theorem 1.1 (Theorem 6.5). *Let $[v] \in \overline{KM}(\mathcal{T})$ be a standard cusp and Z the K3 surface associated to $[v]$ by (*). Then there exists a path $\sigma(t) \in Stab^{\dagger}(\mathcal{T}), t \gg 0$ and an equivalence $\Phi : \mathcal{D}^b(Z) \xrightarrow{\sim} \mathcal{T}$ such that*

1. $\lim_{t\to\infty} \pi(\sigma(t)) = [v] \in \overline{KM}(\mathcal{T})$ *and*
2. $\lim_{t\to\infty} \mathcal{A}(\sigma(t)) = \Phi(Coh(Z))$ *as subcategories of* $\mathcal{T}$.

The path in this theorem is the image of $\sigma_Z(t\beta, t\omega)$ under a certain equivalence. It is easy to construct other paths satisfying (1) which have limiting hearts given by tilts of $Coh(Z)$. The natural question arises how all limiting hearts look like.

Instead of allowing all possible paths we identify a class of paths $\gamma(t) \in KM(\mathcal{T})$, called *linear degenerations to a cusp* $[v] \in \overline{KM}(\mathcal{T})$, and restrict our attention to them. The prototypical example of a linear degeneration is $\pi(\sigma_Y(\beta, t\omega))$. In this case the heart of $\sigma_Y(\beta, t\omega)$ is constant and given by an explicit tilt of $Coh(Y)$. We prove the following proposition.

Proposition 1.2 (Corollary 6.9, Proposition 6.2). *Let $[v] \in \overline{KM}(\mathcal{T})$ be a standard cusp and $\gamma(t) \in KM(\mathcal{T})$ be a linear degeneration to $[v]$, then $\gamma(t)$ is a geodesic converging to $[v]$.*

The (orbifold-)Riemannian metric we use is induced via the isomorphism $\mathfrak{D}(\mathcal{T}) \cong O(2,\rho)/SO(2) \times O(\rho)$.

Conjecture 1.3. *Every geodesic converging to $[v]$ is a linear degeneration.*

This conjecture is true in the case that Y has Picard rank one. Moreover, if one uses the Borel–Serre compactification to compactify $KM(\mathcal{T})$ the conjecture seems to follow from [JM02].

The next theorem classifies paths of stability conditions mapping to linear degenerations in the Kähler moduli space.

Theorem 1.4 (Theorem 6.3). *Let $[v]$ be a standard cusp of $\overline{KM}(\mathcal{T})$. Let $\sigma(t) \in Stab^{\dagger}(Y)$ be a path in the stability manifold such that $\pi(\sigma(t)) \in \overline{KM}(Y)$ is a linear degeneration to $[v]$. Let Z be the K3 surface associated to $[v]$ by (*). Then there exist*

1. *a derived equivalence* $\Phi : \mathcal{D}^b(Z) \xrightarrow{\sim} \mathcal{T}$,
2. *classes* $\beta \in NS(Z)_{\mathbb{R}}$, $\omega \in \overline{Amp}(Z)$ *and*
3. *a path* $g(t) \in \widetilde{Gl}_2^+(\mathbb{R})$

such that

$$\sigma(t) = \Phi_*(\sigma_Z^*(\beta, t\,\omega) \cdot g(t))$$

for all $t \gg 0$.

Moreover, the hearts of $\sigma(t) \cdot g(t)^{-1}$ are independent of t for $t \gg 0$. If $\omega \in Amp(Z)$, then the heart can be explicitly described as a tilt of $Coh(Z)$.

Here, $\sigma_Y^*(\beta, \omega)$ is an extension of Bridgeland's construction of $\sigma_Y(\beta, \omega)$ to the case that $\omega \in \overline{Amp}(Y)$ and $\omega^2 > 2$ (cf. Lemma 4.11).

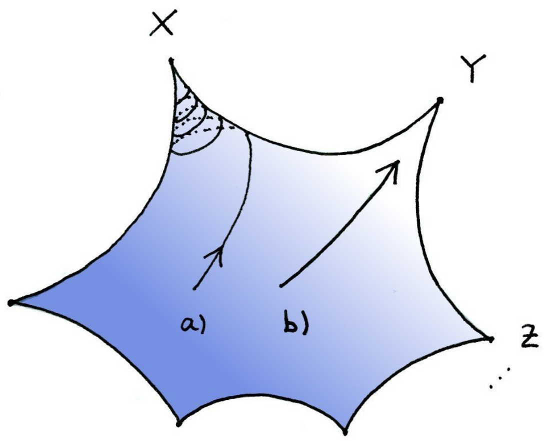

Figure 1.1: Kähler moduli space with cusps, associated K3 surfaces and two different degenerating paths. Paths of type a) are considered in Theorem 6.5, whereas b) pictures a linear degeneration.

Another question we considered is: Can we construct Z as a moduli space of stable objects in stability conditions near the associated cusp $[v] \in \overline{KM}(\mathcal{T})$?

For $v \in N(Y) = N(\mathcal{T})$ and $\sigma \in Stab(Y)$ we consider the following moduli space of semi-stable objects

$$\mathcal{M}^{\sigma}(v) = \left\{ E \in \mathcal{D}^b(Y) \mid E \ \sigma\text{-semi-stable, } v(E) = v \right\} / \sim,$$

where $E \sim F$ if there is an even number $k \in 2\mathbb{Z}$ and a quasi-isomorphism $E \cong F[k]$. This is a version of the moduli stack constructed by Lieblich [Lie06] and Toda [Tod08]. We prove the following result.

Theorem 1.5 (Theorem 7.13)**.** *If $v \in N(Y)$ is an isotropic vector with $v.N(Y) = \mathbb{Z}$ and $\sigma \in Stab^{\dagger}(Y)$ a v-general stability condition, then:*

1. *The moduli space $\mathcal{M}^{\sigma}(v)$ is represented by a K3 surface Z.*

2. *The Hodge structure $H^2(Z, \mathbb{Z})$ is isomorphic to the subquotient of $\widetilde{H}(Y, \mathbb{Z})$ given by $v^{\perp}/\mathbb{Z}v$.*

3. *The universal family $E \in \mathcal{M}^{\sigma}_Y(v)(Z) \subset \mathcal{D}^b(Y \times Z)$ induces a derived equivalence $\mathcal{D}^b(Y) \xrightarrow{\sim} \mathcal{D}^b(Z)$.*

This is in some sense a negative answer to our question: The isomorphism type of $\mathcal{M}^{\sigma}(v)$ does not depend on whether the stability condition σ is close to a cusp or not. On the other hand, the isotropic vector v determines a standard cusp $[v] \in \overline{KM}(\mathcal{T})$ and Z is indeed the K3 surface associated to the cups.

On the way of proving the above result we need to construct enough equivalences that respect the distinguished component $Stab^{\dagger}(\mathcal{T})$. We collected our results in chapter 5 which is essentially independent from the rest of this thesis.

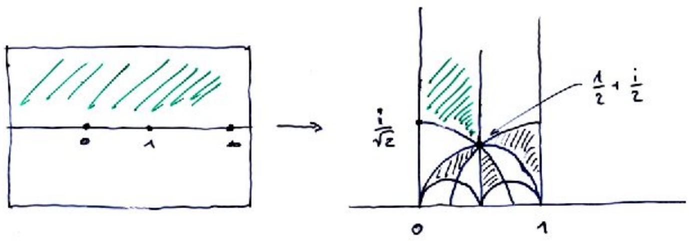

Figure 1.2: Mapping properties of the mirror map ψ in coordinates $z = 1/t^4$ and p.

Theorem 1.6 (Theorem 5.6,Theorem 5.7, Theorem 5.8)**.** *The following equivalences respect the distinguished component.*

- *For a fine, compact, two-dimensional moduli space of Gieseker-stable sheaves $M^h(v)$, the Fourier–Mukai equivalence induced by the universal family.*
- *The spherical twists along Gieseker-stable spherical vector bundles.*
- *The spherical twists along $\mathcal{O}_C(k)$ for a (-2)-curve $C \subset Y$ and $k \in \mathbb{Z}$.*

This allows us to show the following strengthening of a result of [HLOY04], [HMS09].

Proposition 1.7 (Proposition 5.12)**.** *Let $Aut^\dagger(\mathcal{D}^b(Y)) \subset Aut(\mathcal{D}^b(Y))$ be the subgroup of auto-equivalences which respect the distinguished component. Then*

$$Aut^\dagger(\mathcal{D}^b(Y)) \longrightarrow O^+_{Hodge}(\tilde{H}(Y, \mathbb{Z}))$$

is surjective.

Another direct consequence is the description (1.1) of the Kähler moduli space, cf. Corollary 5.13.

1.4 Period- and mirror-maps for the quartic K3

The goal of the second part of this thesis is to construct an explicit pair of K3 surfaces which are mirror dual in the sense of (AM). The property (AM) can be seen as a refinement of Dolgachev's [Dol96] notion of mirror symmetry for families of lattice polarized K3 surfaces (cf. chapter 8.5). There are many examples of mirror dual families of lattice polarized K3 surfaces, e.g. [Bel02], [Roh04], [Dol96]. On the other hand, the author is not aware of an explicit example of mirror symmetry in the Hodge theoretic sense (AM) in the literature.

We study the following families of K3 surfaces.

- Let $Y \subset \mathbb{P}^3$ be a smooth quartic in $\mathbb{P}^3$ viewed as a symplectic manifold with the symplectic structure given by the restriction of the Fubini–Study Kähler form ω_{FS}. We introduce a scaling parameter $p \in \mathbb{H}$ to get a family of (complexified) symplectic manifolds $Y_p = (Y, \omega_p)$, $\omega_p = p/i \cdot \omega_{FS}$ parametrized by the upper half plane.

- Let X_t be the Dwork family of K3 surfaces, which is constructed from the Fermat pencil

$$F_t := \{X_0^4 + X_1^4 + X_2^4 + X_3^4 - 4tX_0X_1X_2X_3 = 0\} \subset \mathbb{P}^3$$

by taking the quotient with respect to a finite group and minimal resolution of singularities.

This is the two-dimensional analogue to the quintic threefold and its mirror studied by Candelas et al. [COGP91].

Theorem 1.8 (Theorem 11.1, Theorem 10.29, Theorem 10.37)**.** *The K3 surfaces X_t and Y_p are mirror dual in the Hodge theoretic sense (AM) if t and p are related by*

$$exp(2\pi ip) = w + 104\,w^2 + 15188\,w^3 + 2585184\,w^4 + 480222434\,w^5 + \dots$$

where $w := 1/(4t)^4$. A closed expression as ratio of hypergeometric functions is given in chapter 10.7.

The multi-valued map $\psi : z \mapsto p(z), z = 1/t^4$ determined by this equation is a Schwarz triangle function which maps the upper half plane to the hyperbolic triangle with vertices $(\infty, \frac{i}{\sqrt{2}}, \frac{1+i}{2})$ and interior angles $(0, \pi/2, \pi/4)$, as pictured in Figure 1.2.

The proof relies heavily on earlier work by Narumiyah and Shiga [NS01], Dolgachev [Dol96] and Nagua and Sugiyama [NS95]. We proceed in three main steps: First, we use a theorem of Narumiyah and Shiga which provides us with the required cycles and a description of the topological monodromy of the family. Then we consider the Picard–Fuchs differential equation which is satisfied by the period integrals. We derive a criterion for a set of solutions to be the coefficients of the period map. In a third step we construct solutions to this differential equation which match this criterion. Here we use the work of Nagura and Sugiyama. The relation to Schwarz triangle function also appears in [NS01, Thm. 6.1].

The function in Theorem 1.8 was also considered by Lian and Yau [LY96] (see Remark 11.2). There it was noted that the inverse function $z(p)$ is a modular form with integral Fourier expansion which is related to the Thompson series for the Griess–Fischer ("monster") group. See also the exposition by Verrill and Yui in [VY00].

Our motivation for studying this specific family stems from a theorem of Seidel. He proves homological mirror symmetry (HMS) for the pair of K3 surfaces considered above. In fact, apart from the case of elliptic curves (Polishchuk–Zaslow [PZ98]) and some progress for abelian varieties (Kontsevich–Soibelman [KS01], Fukaya [Fuk02]) there is no other example of compact Calabi–Yau manifolds where (HMS) is known to hold.

Theorem (Seidel [Sei03])**.** *If the family X_t is viewed as a K3 surface X over the Novikov field $\Lambda_{\mathbb{Q}}(1/t)$, which is the algebraic closure of the field of formal Laurent series $\mathbb{C}((1/t))$, then there is an isomorphism $\psi : \Lambda_{\mathbb{Q}}(1/t) \cong \Lambda_{\mathbb{Q}}(p)$ and an equivalence of triangulated $\Lambda_{\mathbb{Q}}(p)$-linear categories*

$$\psi_* \mathcal{D}^b(Coh(X)) \cong \mathcal{D}^\pi(Fuk(Y)).$$

Unfortunately, the isomorphism ψ has not yet been determined explicitly. Geometrically it describes the dependence of the symplectic volume p of the quartic on the deformation parameter t of the complex structure on X. Thus our mirror map ψ in Theorem 1.8 provides a conjectural candidate for this isomorphism.

On the way to proving Theorem 1.8 we also give an explicit calculation of the classical period map for the Dwork family. Consider a non-zero holomorphic two-form $\Omega \in H^{2,0}(X)$ and a basis of two-dimensional cycles $\Gamma_i \in H_2(X, \mathbb{Z}) \cong \mathbb{Z}^{22}$. By the global Torelli theorem, the complex structure on X is determined by the period integrals $(\int_{\Gamma_1} \Omega, \ldots, \int_{\Gamma_{22}} \Omega)$ and the intersection numbers $\Gamma_i.\Gamma_j$.

Theorem 1.9 (Theorem 10.29, Remark 10.32)**.** *For $t \in \mathbb{C}$ near $t_0 = i/\sqrt{2}$, there are explicit bases $\Gamma_i(t) \in H_2(X_t, \mathbb{Z}), i = 1, \ldots, 22$ and holomorphic two-forms Ω_t on the Dwork family X_t such that the period integrals are given by*

$$(\int_{\Gamma_1(t)} \Omega_t, \ldots, \int_{\Gamma_{22}(t)} \Omega_t) = (4p(t), 2p(t)^2, -1, p(t), 0, \ldots, 0),$$

where $p(t) = \psi(1/t^4)$ is the function introduced in Theorem 1.8.

1.5 Acknowledgments

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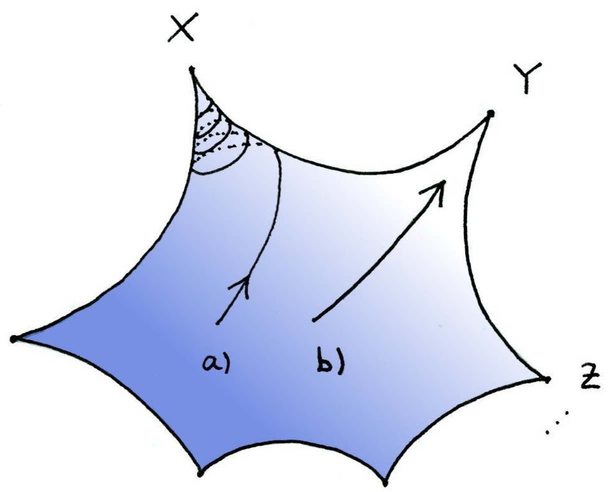
Part I

Cusps of the Kähler moduli space

and

stability conditions on K3 surfaces



Notation

Our notation will largely follow Huybrechts' book [Huy06]. A K3 surface is a connected projective surface X over the complex numbers with trivial canonical bundle $\Omega_X^2 \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$. The Picard rank of X is denoted by $\rho(X) = rk(NS(X))$.

We write $\tilde{H}(X, \mathbb{Z})$ for the full cohomology $H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$ endowed with the Mukai pairing $(r, l, s).(r', l', s') = l.l' - rs' - r's$, and the weight-two Hodge structure

$$\tilde{H}^{1,1}(X) = H^{0,0}(X) \oplus H^{1,1}(X) \oplus H^{2,2}(X),$$

$$\tilde{H}^{2,0}(X) = H^{2,0}(X), \quad \tilde{H}^{0,2}(X) = H^{0,2}(X).$$

We write $N(X) = H^0(X, \mathbb{Z}) \oplus NS(X) \oplus H^4(X, \mathbb{Z})$ for the extended Néron–Severi group. It is an even lattice of signature $(2, \rho(X))$.

To a sheaf $A \in Coh(X)$ we associate the Mukai vector

$$v(A) = \sqrt{\mathrm{td}(X)}.ch(A) = (r(A), c_1(A), s(A)) \in N(X),$$

where $s(A) = \frac{1}{2}c_1(A)^2 - c_2(A) + r(A)$. By the Riemann–Roch theorem we have $-\chi(A, B) = v(A).v(B)$. Therefore, we can identify $N(X)$ with the numerical Grothendieck group $N(Coh(X)) = K(Coh(X))/\mathrm{rad}(\chi)$ via the map $A \mapsto v(A)$.

We denote by $\mathcal{D}^b(X)$ the bounded derived category of coherent sheaves on X. We have natural isomorphisms between the numerical Grothendieck groups $N(\mathcal{D}^b(X)) \cong N(Coh(X)) \cong N(X)$. Every $\mathbb{C}$-linear, exact equivalence $\Phi : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ induces a Hodge isometry which we denote by

$$\Phi^H : \tilde{H}(X, \mathbb{Z}) \longrightarrow \tilde{H}(Y, \mathbb{Z}).$$

We say that two K3 surfaces X and Y are derived equivalent if $\mathcal{D}^b(X)$ is equivalent to $\mathcal{D}^b(Y)$ as a $\mathbb{C}$-linear, triangulated category.

Chapter 2

Geometry of the Mukai lattice

Let X be a K3 surface and $\mathcal{D}^b(X)$ its derived category of coherent sheaves. Let $N = N(\mathcal{D}^b(X))$ be the numerical Grothendieck group of $\mathcal{D}^b(X)$. In this chapter we will introduce various groups and spaces that are naturally associated to the lattice N.

The isomorphism $N \cong N(X)$ gives us the following extra structures.

1. An isotropic vector $v_0 = (0, 0, 1) \in N$.
2. An embedding of a hyperbolic plane
$$\varphi : U \cong H^0(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}) \to N.$$
3. The choice of an ample chamber $Amp(X) \subset NS(X)_{\mathbb{R}} \subset N_{\mathbb{R}}$.
4. A weight-two Hodge structure $\tilde{H}(X, \mathbb{Z})$ with $\tilde{H}^{1,1}(X) \cap \tilde{H}(X, \mathbb{Z}) = N$. In particular, a group action of $O_{Hodge}(\tilde{H}(X, \mathbb{Z}))$ on N.

We will pay special attention to which constructions depend on what additional data.

Convention 2.1. In later chapters, when we have fixed an identification $N = N(X)$, we will allow ourselves to abuse the notation by filling in the standard choices of the above extra structures. For example, we shall write $\mathcal{L}(X)$ for the space $\mathcal{L}(N(X), v_0, Amp(X))$ introduced in Definition 2.13.

2.1 The Kähler period domain

Let N be a non-degenerate lattice of signature $(2, \rho)$.

Definition 2.2. We define the *Kähler period domain* to be

$$\mathfrak{D}(N) = \left\{ [z] \in \mathbb{P}(N_{\mathbb{C}}) \mid z^2 = 0,\ z.\bar{z} > 0 \right\} \subset \mathbb{P}(N_{\mathbb{C}}).$$

We also introduce the following open subset of $N_{\mathbb{C}}$:

$$\mathcal{P}(N) = \{ z \in N_{\mathbb{C}} \mid \mathbb{R}\langle Re(z), Im(z)\rangle \subset N_{\mathbb{R}} \text{ is a positive 2-plane} \} \subset N_{\mathbb{C}}.$$

This set carries a natural free $Gl_2(\mathbb{R})$-action by identifying $N_{\mathbb{C}} = N \otimes_{\mathbb{Z}} \mathbb{C}$ with $N \otimes_{\mathbb{Z}} \mathbb{R}^2$.

Lemma 2.3. *There is a canonical map*

$$\theta : \mathcal{P}(N) \longrightarrow \mathfrak{D}(N)$$

which is a principal $Gl_2^+(\mathbb{R})_2^+$-bundle.

Proof. This map is most easily described using the canonical isomorphism between $\mathfrak{D}(N)$ and the Grassmann manifold $Gr_2^{po}(N_{\mathbb{R}})$ of positive definite, oriented two-planes in $N_{\mathbb{R}}$ (cf. [BBD85, VII. Lem.1]). We define θ to map a vector $[z] \in \mathcal{P}(N)$ to the oriented two-plane $P = \mathbb{R}\langle Re(z), Im(z)\rangle$. As $Gl_2^+(\mathbb{R})$ acts simply and transitively on the set of oriented bases of P, this map is a principal $Gl_2^+(\mathbb{R})$-bundle. □

In the case $N = N(X) = H^0(X) \oplus NS(X) \oplus H^4(X)$ there is a well known *tube model* of the period domain, given by

$$exp\colon \{z = x + iy \in NS(X)_{\mathbb{C}} \,|\, y^2 > 0\} \xrightarrow{\cong} \mathfrak{D}(N),\ z \mapsto [(1, z, \frac{1}{2}z^2)].$$

To define this map we used the full information about the embedding $U \cong H^0(X) \oplus H^4(X)$ into $N(X)$. In this chapter we will construct a similar map, which only depends on the isotropic vector $v_0 = (0, 0, 1)$. Compare also [Dol96, Sec. 4].

Let N be a non-degenerate lattice of signature $(2, \rho), \rho \geq 1$. To a primitive isotropic vector $v \in N$ we associate the lattice

$$L(v) = v^{\perp}/\mathbb{Z}v = \{\, z \in N \mid z.v = 0 \,\}/\mathbb{Z}v$$

of signature $(1, \rho - 1)$. If moreover $v.N = \mathbb{Z}$, we define an affine space

$$A(v) = \{\, z \in N \mid z.v = -1 \,\}/\mathbb{Z}v$$

over $L(v)$. The real variant $A(v)_{\mathbb{R}} = \{\, z \in N_{\mathbb{R}} \mid z.v = -1 \,\}/\mathbb{R}v$ is defined for arbitrary isotropic vectors $v \in N_{\mathbb{R}}$ and an affine space over $L(v)_{\mathbb{R}}$. Note that, if $N = N(X)$ and $v = v_0$, then $L(v) \cong NS(X)$.

Definition 2.4. We define the *tube domain* associated to N and v as

$$T(N, v) = A(v)_{\mathbb{R}} \times C(L(v))$$

where $C(L(v)) = \left\{\, y \in L(v)_{\mathbb{R}} \mid y^2 > 0 \,\right\}$. We consider $T(N, v)$ as a subset of $N_{\mathbb{C}}/\mathbb{C}v$ by mapping (x, y) to $x + iy \in N_{\mathbb{C}}/\mathbb{C}v$. We will often write $x + iy$ for a pair $(x, y) \in T(N, v)$.

Lemma 2.5. *There is a canonical map $Exp_v : T(N, v) \to \mathcal{P}(N)$ such that*

$$exp_v = \theta \circ Exp_v : T(N, v) \longrightarrow \mathfrak{D}(N)$$

is an isomorphism.

Proof. We construct the inverse to Exp_v. The set

$$Q(v) = \left\{ z \in N_{\mathbb{C}} \mid z^2 = 0, z.\bar{z} > 0, z.v = -1 \right\} \subset \mathcal{P}(N)$$

is a section for the $Gl_2^+(\mathbb{R})$-action on $\mathcal{P}(N)$.

Indeed, let $(P, o) \in Gr^{po}(N_{\mathbb{R}})$ be a positive, oriented two-plane. If $x \in P$ is a vector, then there is a unique $y \in P$ such that x, y is an oriented orthogonal basis with $x^2 = y^2$. Therefore $z = x + iy$ satisfies $z^2 = 0$ and $z.\bar{z} = 2y^2 > 0$. It is easy to see, that the map $P \to \mathbb{C}, x \mapsto (x.v) + i(y.v)$ is an isomorphism. Furthermore, x maps to -1 if and only if $z = x + iy$ lies in $Q(v)$.

One checks immediately, that the projection $N_{\mathbb{C}} \to N_{\mathbb{C}}/\mathbb{C}v$ induces an isomorphism $Q(v) \to T(N, v)$. Define Exp_v to be the inverse of this isomorphism. □

Remark 2.6. In particular, we obtain a section of the $Gl_2^+(\mathbb{R})$-bundle θ, namely $q_v = Exp_v \circ exp_v^{-1} : \mathfrak{D}(N) \to \mathcal{P}(N)$.

Lemma 2.7. *Let $g \in O(N)$ be an isometry of N, then g induces a commutative diagram*

$$\begin{tikzcd} T(N,v) \arrow[r, "Exp_v"] \arrow[d, "g"'] & \mathcal{P}(N) \arrow[r, "\theta"] \arrow[d, "g"] & \mathfrak{D}(N) \arrow[d, "g"] \\ T(N,w) \arrow[r, "Exp_w"] & \mathcal{P}(N) \arrow[r, "\theta"] & \mathfrak{D}(N), \end{tikzcd}$$

where $w = g \cdot v$.

Proof. We use the notation from the proof of Lemma 2.5. The element g induces an isomorphism $Q(v) \to Q(w)$, as well as $N_{\mathbb{C}}/\mathbb{C}v \to N_{\mathbb{C}}/\mathbb{C}w$. The projection $Q(v) \to N_{\mathbb{C}}/\mathbb{C}v$ commutes with the action of g. Therefore Exp_v has the same property. The equivariance of θ is trivial. □

2.2 Roots, Walls and Chambers

To a lattice N we associate a *root-system* $\Delta(N) = \left\{ \delta \in N \mid \delta^2 = -2 \right\}$. To every root $\delta \in \Delta(N)$ there is an associated *reflection*, $s_\delta : w \mapsto w + (\delta.w)\delta$ which is an involutive isometry. The subgroup $W(N) \subset O(N)$ generated by the reflections is called *Weyl group*.

If we are given an isotropic vector $v \in N$ we define

$$\Delta^{>0}(N, v) = \{ \delta \in \Delta(N) \mid -v.\delta > 0 \}, \quad \Delta^0(N, v) = \{ \delta \in \Delta(N) \mid v.\delta = 0 \}.$$

Note that, $\Delta(N)$ is the disjoint union of $\Delta^0(N, v)$, and $\pm\Delta^{>0}(N, v)$. The group generated by the reflections $\left\{ s_\delta \mid \delta \in \Delta^0(N, v) \right\}$ is denoted by $W^0(N, v)$.

Definition 2.8. To $\delta \in \Delta(N)$ we associate a divisor

$$D(\delta) = \{ [z] \mid z.\delta = 0 \} \subset \mathfrak{D}(N)$$

and define $\mathfrak{D}_0(N) = \mathfrak{D}(N) \setminus \bigcup \{ D(\delta) \mid \delta \in \Delta(N) \}$.

The connected components of $\mathfrak{D}(N) \cong T(N,v)$ are clearly contractible. In contrast the space $\mathfrak{D}_0(N)$ is the complement of a infinite number of hypersurfaces and will therefore in general not even have a finitely generated fundamental group. Following Bridgeland (cf. Remark 4.17) we will decompose $\mathfrak{D}_0(N)$ into a union of codimension-one submanifolds called *walls* and their complements called *chambers* such that each individual chamber is contractible.

Definition 2.9. Given $\delta \in \Delta^{>0}(N,v)$ and a primitive, isotropic vector $v \in N$, we define a real, codimension one submanifold, called *wall*

$$W_A(\delta,v) = \{ [z] \in \mathfrak{D}(N) \mid -z.\delta/z.v \in \mathbb{R}_{\leq 0} \} \subset \mathfrak{D}(N).$$

To a vector $\delta \in \Delta^0(N,v)$ we associate the wall

$$W_C(\delta,v) = \{ [z] \in \mathfrak{D}(N) \mid -z.\delta/z.v \in \mathbb{R} \}.$$

One can check, that $W_C(\delta,v)$ only depends on the image l of δ in $L(v) = v^{\perp}/\mathbb{Z}v$. Therefore we write also $W_C(l,v)$ for $W_C(\delta,v)$. We define

$$\begin{aligned} \mathfrak{D}_A(N,v) &= \mathfrak{D}(N) \setminus \bigcup \{ W_A(\delta,v) \mid \delta \in \Delta^{>0}(N,v) \} \\ \mathfrak{D}_C(N,v) &= \mathfrak{D}(N) \setminus \bigcup \{ W_C(\delta,v) \mid \delta \in \Delta^{0}(N,v) \}. \end{aligned}$$

We denote the intersections $\mathfrak{D}_0(N) \cap \mathfrak{D}_A(N,v), \mathfrak{D}_A(N,v) \cap \mathfrak{D}_C(N,v)$, etc. by $\mathfrak{D}_{0,A}(N,v), \mathfrak{D}_{A,C}(N,v)$, etc., respectively. For any combination $*$ of the symbols $0, A, C$ we set

$$T_*(N,v) = exp_v^{-1}(\mathfrak{D}_*(N,v)).$$

Remark 2.10. The seemingly unnatural notation, $-z.\delta/z.v \in \mathbb{R}_{\leq 0}$, is chosen since for $z = Exp_v(x+iy)$ we have $z.v = -1$, and hence $-z.\delta/z.v = z.\delta$.

The sets considered above are indeed complex manifolds as the unions of $D(\delta), W_A(\delta,v)$ and $W_C(l,v)$ are locally finite [Bri08, Lem. 11.1].

If $\delta \in \Delta^0(N,v)$, then $D(\delta) \subset W_C(\delta,v)$, and if $\delta \in \pm\Delta^{>0}(N,v)$, then $D(\delta) \subset W_A(\delta,v)$. Therefore,

$$\mathfrak{D}_{A,C}(N,v) \subset \mathfrak{D}_0(N,v),$$

and $\mathfrak{D}_{0,A,C}(N,v) = \mathfrak{D}_{A,C}(N,v)$.

Lemma 2.11. *[Bri08, Lem. 6.2, Lem. 11.1] Let*

$$\mathfrak{D}_{>2}(N,v) = \left\{ exp_v(x+iy) \in \mathfrak{D}(N) \mid y^2 > 2 \right\} \subset \mathfrak{D}(N)$$

and denote by $\mathfrak{D}_{0,>2}(N,v) = \mathfrak{D}_{>2}(N,v) \cap \mathfrak{D}_0(N,v)$ etc. the various intersections. Then

$$\mathfrak{D}_{>2}(N,v) \subset \mathfrak{D}_A(N,v)$$

and moreover the inclusions $\mathfrak{D}_{>2}(N,v) \subset \mathfrak{D}_A(N,v)$, $\mathfrak{D}_{0,>2}(N,v) \subset \mathfrak{D}_{0,A}(N,v)$ and $\mathfrak{D}_{0,C,>2}(N,v) \subset \mathfrak{D}_{0,C,A}(N,v)$ are deformation retracts.

Proof. For convenience of the reader we repeat the argument. Let $z = Exp_v(x+iy) \in \mathcal{P}_A(N,v)$ be a lift of $[z] \in \mathfrak{D}_A(N,v)$. We have to show that for $\delta \in \Delta^{>0}(N,v)$ the inequality $\delta.z \notin \mathbb{R}_{\leq 0}$ holds. Suppose that $\delta.z \in \mathbb{R}_{\leq 0}$. Write

$\delta = (r, l, s)_\varphi$, then $0 = Im(\delta.z) = (l - rx).y$. As $y^\perp$ is a negative lattice it follows that $(l - rx)^2 < 0$. Now, the assumption $y^2 > 2$ implies that

$$Re(\delta.z) = \frac{1}{2r}(-2 + r^2y^2 - (l - rx)^2) > 0,$$

which is in contradiction to $\delta.z \in \mathbb{R}_{\leq 0}$.

For the second part let

$$z(t) = Exp_v(x + ity), \quad t \geq 1.$$

If $z \in \mathcal{P}_A(N, v)$, then $z(t) \in \mathcal{P}_A(N, v)$ for all $t \geq 1$. This follows immediately from the formulas for $Re(z(t).\delta)$ and $Im(z(t).\delta)$ given above. Now, $z(t) \in \mathcal{P}_{>2}(N)$ as soon as $t^2 > 1/y^2$. Therefore, we can retract $\mathcal{P}_A(N)$ along the paths $z(t), t \geq 1$ into $\mathcal{P}_{>2}(N)$. The same argument applies also for $\mathcal{P}_{0,>2}(N, v)$ and $\mathcal{P}_{0,C,>2}(N, v)$. □

Recall that, $L = L(v)$ is a lattice of signature $(1, \rho - 1)$. We defined $C(L) = \left\{ y \in L_\mathbb{R} \mid y^2 > 0 \right\}$ to be the *positive cone.* This space has two connected components, let $C(L)^+$ be one of them. For $l \in \Delta(L)$ define a *wall* $W(l) = \{ y \in C(L) \mid y.l = 0 \}$ and set $C_0(L) = C(L) \setminus \bigcup \{ W(l) \mid l \in \Delta(L) \}$. Connected components of $C(L)_0$ are called *chambers.*

Lemma 2.12. *We have*

$$T_C(N, v) = A(v)_\mathbb{R} \times C_0(L(v)).$$

Moreover, the connected components of $\mathfrak{D}_{0,A,C}(N, v) \subset \mathfrak{D}_0(N)$ *are contractible.*

Proof. Assume that $exp_v(z) \in W_C(\delta, v) \subset \mathfrak{D}(N)$ for some $\delta \in \Delta^0(N, v)$. Let $l \in L(v)$ be the image of δ under the canonical projection $N \supset v^\perp \to L(v)$. Note that, $l^2 = -2$. We have $Exp_v(z).\delta \in \mathbb{R}$ and writing out the definition one sees that $Im(z).l = 0$.

Conversely, if $Im(z).l = 0$ for some $l \in \Delta(N)$, every lift $\delta \in N$ of $l \in L(v)$ lies in $\Delta^0(N, v)$. One verifies easily, that $exp_v(z).\delta \in \mathbb{R}$.

Let $Amp \subset C(L)_0$ be a connected component. We claim that Amp is convex, and hence contractible. Let $y, y' \in Amp$ and $l \in \Delta(L)$ then $y.l > 0$ or $y.l < 0$. If $y.l > 0$ then $y'.l > 0$ as well, since Amp is connected. Hence, in this case also $((1 - t)y + ty').l > 0$ for all $0 \leq t \leq 1$. We argue similarly if $y.l < 0$. This shows the claim.

The connected components of $\mathfrak{D}_{C,>2}(N, v)$ are contractible since the inclusion $\mathfrak{D}_{C,>2}(N, v) \subset \mathfrak{D}_C(N, v)$ is a deformation retract (cf. Lemma 2.11).

Also, $\mathfrak{D}_{C,>2}(N, v) \subset \mathfrak{D}_{0,A,C}(N, v)$ is a deformation retract, and hence the components of $\mathfrak{D}_{0,A,C}(N, v)$ are contractible. □

Definition 2.13. If we are given a chamber $Amp \subset C(L(v))_0$ we define

$$\mathcal{L}(N, v, Amp) \subset \mathfrak{D}_{0,A,C}(N, v)$$

to be the connected component containing the vectors $exp_v(x + iy)$ with $y \in Amp, y^2 > 2$.

The connected component of $\mathfrak{D}(N)$ containing $\mathcal{L}(N, v, Amp)$ is denoted by $\mathfrak{D}^+(N)$. We introduce also the notation $\mathfrak{D}^+_*(N) = \mathfrak{D}^+(N) \cap \mathfrak{D}_*(N, v)$ for a combination $*$ of the symbols $0, A, C, > 2$.

The orthogonal group $O(N)$ acts on $\mathfrak{D}(N)$. We denote by $O^+(N)$ the index two subgroup preserving the connected components of $\mathfrak{D}(N)$.

Remark 2.14. The set $\mathcal{L}(N, v, Amp)$ can be described more explicitly as

$$\{\, exp(x+iy) \in \mathfrak{D}(X) \mid y \in Amp, (*) \,\}$$

where $(*)$ is the condition

$$Exp(x+iy).\delta \notin \mathbb{R}_{\leq 0} \quad \text{for all} \quad \delta \in \Delta^{>0}(N(X), v).$$

This is the description used in [Bri08].

Chapter 3

Ma's Theorem

The goal of this chapter is to explain Ma's theorem about cusps of the Kähler moduli space of a K3 surface ([Ma09], [Ma10]).

We use the recent result [HMS09] to make the construction of the Kähler moduli space intrinsic to the derived category. This allows us to formulate Ma's theorem in a more symmetric way.

3.1 The Kähler moduli space

Recall from Definition 2.13, that $\mathfrak{D}(X) = \mathfrak{D}(N(X))$ has a distinguished connected component $\mathfrak{D}^+(X)$ containing the vectors $exp(x+iy)$ with $y \in NS(X)$ ample. The key ingredient for our construction of $KM(\mathcal{T})$ is the following theorem.

Theorem 3.1. *[HLOY04],[Plo05], [HMS09, Cor. 4.10] The image of*

$$Aut(\mathcal{D}^b(X)) \longrightarrow O_{Hodge}(\tilde{H}(X,\mathbb{Z}))$$

is the index-two subgroup $O^+_{Hodge}(\tilde{H}(X,\mathbb{Z}))$ of isometries preserving the component $\mathfrak{D}^+(X) \subset \mathfrak{D}(X)$.

Let $\Phi : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ be a derived equivalence between two K3 surfaces. Then the isomorphism $\Phi^H : \mathfrak{D}(X) \to \mathfrak{D}(Y)$ maps $\mathfrak{D}^+(X)$ to $\mathfrak{D}^+(Y)$.

This theorem allows us to make the following definition.

Definition 3.2. Let $\mathfrak{D}^+(\mathcal{T})$ be the connected component of $\mathfrak{D}(\mathcal{T}) = \mathfrak{D}(N(\mathcal{T}))$ which is mapped to $\mathfrak{D}^+(X)$ under every derived equivalence $\mathcal{T} \cong \mathcal{D}^b(X)$.

We define the *Kähler moduli space* of $\mathcal{T}$ to be

$$KM(\mathcal{T}) = \Gamma_{\mathcal{T}} \setminus \mathfrak{D}^+(\mathcal{T})$$

where $\Gamma_{\mathcal{T}}$ is the image of $Aut(\mathcal{T})$ in $O(N(\mathcal{T}))$.

Remark 3.3. Let us introduce the notation $KM(X)$ for $KM(\mathcal{D}^b(X))$. Theorem 3.1 shows, that we have a canonical isomorphism

$$KM(X) \cong \Gamma^+_X \setminus \mathfrak{D}^+(X),$$

where $\Gamma_X^+ \subset O(N(X))$ is the image of $O^+_{Hodge}(\tilde{H}(X,\mathbb{Z}))$ in $O(N(X))$. Ma works in the setting $\mathcal{T} = \mathcal{D}^b(X)$ and uses $\Gamma_X^+ \setminus \mathfrak{D}^+(X)$ as definition for the Kähler moduli space.

Remark 3.4. There is another construction of the Kähler moduli space using the theory of Bridgeland stability conditions which is proved in Corollary 5.13:

$$KM_0(X) \cong Aut^\dagger(\mathcal{D}^b(X)) \setminus Stab^\dagger(X)/\widetilde{Gl}_2^+(\mathbb{R}).$$

Here $KM_0(X) = \Gamma_X^+ \setminus \mathfrak{D}_0^+(X) \subset KM(X)$ is the complement of a divisor and $Aut^\dagger(\mathcal{D}^b(X))$ is the group of auto-equivalences respecting the distinguished component $Stab^\dagger(X)$ of the stability manifold. This was also stated in [Ma09] without proof.

Note that, this description is not intrinsic to the derived category as the component $Stab^\dagger(X) \subset Stab(\mathcal{D}^b(X))$ may a priori depend on X. But in fact, no other component of the stability manifold is known.

Example 3.5. If X has Picard rank $\rho(X) = 1$ and the ample generator $H \in NS(X)$ has square $H.H = 2n$, then the Kähler moduli space is isomorphic to a Fricke modular curve $KM(X) \cong \Gamma_0^+(n) \setminus \mathbb{H}$. See [Ma10, Sec. 5], [Dol96, Thm. 7.1].

The subgroup $\Gamma_{\mathcal{T}} \subset O(N(\mathcal{T}))$ is of finite index since it contains

$$O_0^+(N(\mathcal{T})) = O^+(N(\mathcal{T})) \cap Ker(O(N(\mathcal{T})) \to Aut(A(N(\mathcal{T}))))$$

where $A(N(\mathcal{T})) = N(\mathcal{T})^\vee/N(\mathcal{T})$ is the discriminant group, cf. [Ma09, Def. 3.1.]. Hence we can apply a general construction of Baily and Borel to compactify the Kähler moduli space.

Theorem 3.6 (Baily–Borel)**.** *[BB66] There exists a canonical compactification $\overline{KM}(\mathcal{T})$ of $KM(\mathcal{T})$ which is a normal, projective variety over $\mathbb{C}$.*

The boundary $\partial KM(\mathcal{T}) = \overline{KM}(\mathcal{T}) \setminus KM(\mathcal{T})$ consists of zero- and one-dimensional components called cusps, *which are in bijection to $\Gamma_{\mathcal{T}} \setminus \mathcal{B}_i$, where*

$$\mathcal{B}_i = \{\, I \subset N(\mathcal{T}) \mid I \text{ primitive, isotropic, } rk(I) = i+1 \,\}$$

for $i = 0, 1$ respectively.

Definition 3.7. The set of zero-dimensional cusps is divided further with respect to divisibility. For $I \in \mathcal{B}_0$ we define

$$div(I) = g.c.d(\{\, v.w \mid v \in I, w \in N(\mathcal{T}) \,\})$$

and set $\mathcal{B}_0^d = \{\, I \in \mathcal{B}_0 \mid div(I) = d \,\}$. Cusps corresponding to elements of $\mathcal{B}_0^1$ are called *standard cusps.*

We call $v \in N$ a *standard vector*[1] if $v.v = 0$ and $div(v) := div(\mathbb{Z}v) = 1$.

Remark 3.8. The group $\Gamma_{\mathcal{T}}$ contains the element $-id_{N(\mathcal{T})} = [1]^H$ which interchanges the generators of any $I \in \mathcal{B}_0^1$. Therefore, the map $v \mapsto \mathbb{Z}v$ induces a bijection

$$\Gamma_{\mathcal{T}} \setminus \{\, v \in N(\mathcal{T}) \mid v \text{ standard } \} \cong \{ \text{ standard cusps of } \overline{KM}(\mathcal{T}) \ \}.$$

We will refer to standard cusps as equivalence classes $[v] = \Gamma_{\mathcal{T}} \cdot v$ via this bijection.

[1]This definition is not standard.

3.2 Ma's theorem

Definition 3.9. The Kähler moduli space of $\mathcal{D}^b(X)$ comes with a distinguished standard cusp $[v_0] \in \overline{KM}(X), v_0 = (0,0,1) \in N(X)$, which is called *large volume limit.*

We are now ready to state Ma's theorem.

Theorem 3.10 (Ma)**.** *[Ma09], [Ma10] There is a canonical bijection*

$$\left\{ X \text{ K3 surface } \mid \mathcal{T} \cong \mathcal{D}^b(X) \right\}_{/\cong} \longleftrightarrow \{ \text{ standard cusps of } \overline{KM}(\mathcal{T})\}.$$

The cusp of $\overline{KM}(\mathcal{T})$ associated to Y corresponds to the large volume limit of X under the isomorphism $\overline{KM}(\mathcal{T}) \cong \overline{KM}(X)$ induced by any equivalence $\mathcal{T} \cong \mathcal{D}^b(X)$.

We denote the K3 surface associated to a cusp $[v]$ by $X(v)$.

Proof. We sketch Ma's original proof for the case $\mathcal{T} = \mathcal{D}^b(X)$ and then generalize to our situation.

Every derived equivalence $\Phi : \mathcal{D}^b(Y) \to \mathcal{D}^b(X)$ induces an isometry $\Phi^H : N(Y) \to N(X)$, and therefore an embedding of the hyperbolic plane

$$U \cong H^0(Y) \oplus H^4(Y) \subset N(Y) \xrightarrow{\Phi^H} N(X).$$

It follows from Orlov's derived global Torelli theorem [Huy06, Prop. 10.10] that this construction induces a bijection

$$\left\{ Y \text{ K3 surface} \mid \mathcal{D}^b(X) \cong \mathcal{D}^b(Y) \right\}_{/\cong} \longrightarrow Emb(U, N(X))/\Gamma_X,$$

where $Emb(U, N(X))$ is the set of all embeddings of the hyperbolic plane U into $N(X)$, and $\Gamma_X \subset O(N(X))$ is the image of $O_{Hodge}(\tilde{H}(X,\mathbb{Z}))$ in $O(N(X))$. The key insight of Ma is that the map $\varphi \mapsto \varphi(f)$, where $e, f \in U$ is the standard basis, induces a bijection

$$Emb(U, N(X))/\Gamma_X \longrightarrow \{ v \in N(X) \mid v \text{ standard } \}/\Gamma_X^+.$$

Combining with Remark 3.8 one gets a bijection

$$\left\{ Y \text{ K3 surface} \mid \mathcal{D}^b(X) \cong \mathcal{D}^b(Y) \right\}_{/\cong} \longrightarrow \{ \text{ standard cusps of } \overline{KM}(X) \}$$

which maps X maps to $[v_0]$. Note that, the Hodge structure $H^2(Y,\mathbb{Z})$ of a K3 surface Y can be reconstructed from the associated cusp $[v]$ as the subquotient $v^\perp/v$ of $\tilde{H}(X,\mathbb{Z})$.

To generalize to arbitrary $\mathcal{T}$ we choose an equivalence $\mathcal{T} \cong \mathcal{D}^b(X)$ and claim that the above bijection is independent of this choice. Indeed, if we are given another equivalence $\mathcal{T} \cong \mathcal{D}^b(Y)$, then the composition $\Phi : \mathcal{D}^b(X) \cong \mathcal{T} \cong \mathcal{D}^b(Y)$ induces a Hodge isometry $\Phi^H : \tilde{H}(X,\mathbb{Z}) \to \tilde{H}(Y,\mathbb{Z})$. If a standard vector $v \in N(\mathcal{T})$ corresponds to $v_1 \in N(X)$ and $v_2 \in N(Y)$ then Φ^H induces an isomorphism of Hodge structures

$$H^2(X(v_1),\mathbb{Z}) \cong v_1^\perp/v_1 \longrightarrow v_2^\perp/v_2 \cong H^2(X(v_2),\mathbb{Z}).$$

Now, the global Torelli theorem shows that the K3 surfaces $X(v_1)$ and $X(v_2)$ are isomorphic. ∎

Remark 3.11. Let $v \in N(X)$ be a standard vector defining a standard cusp of $\overline{KM}(X)$. The Fourier–Mukai partner $Y = X(v)$ associated to this cusp via Ma's theorem is determined up to isomorphism, by the property that

$$H^2(X(v), \mathbb{Z}) \cong v^\perp / v$$

as subquotient of $\tilde{H}(X, \mathbb{Z})$.

We cannot formulate an analogues statement for the cusps of $\overline{KM}(\mathcal{T})$ since there is no construction of the Hodge structure $\tilde{H}(X, \mathbb{Z})$ known, which is intrinsic to the category $\mathcal{D}^b(X)$.

Remark 3.12. Let X be a K3 surface, $[v] \in \overline{KM}(X)$ a standard cusp and let $Y = X(v)$ be the associated Fourier–Mukai partner. We have seen that every derived equivalence $\Phi : \mathcal{D}^b(X) \xrightarrow{\sim} \mathcal{D}^b(Y)$ maps $[v]$ to the large volume limit $[v_0] \in \overline{KM}(Y)$.

In chapter 5.4 we will strengthen this result in two directions. Firstly, we will construct a Φ, with the property that Φ^H maps v to v_0 and not only the orbit $[v]$ to $[v_0]$. Secondly, the equivalence Φ respects the distinguished component of the stability manifold (cf. Section 4.3).

Chapter 4

Stability conditions

Our next goal is to relate the Kähler moduli space to the stability manifold. In this chapter we recall from [Bri07] and [Bri08] the basic theory of Bridgeland stability conditions in the special case of a K3 surface. On the way we introduce the notation and establish some geometric results that will be used in sequel. The link to the Kähler moduli space will be made in chapter 6 and Corollary 5.13.

4.1 Definition of stability conditions

Let X be a K3 surface. Recall from [Bri07, Def. 5.7, Def. 2.3., Prop. 5.3], that a *stability condition* σ on $\mathcal{D}^b(X)$ consists of

1. a heart $\mathcal{A}$ of a bounded t-structure on $\mathcal{D}^b(X)$ and
2. a vector $z \in N(X)_{\mathbb{C}}$ called *central charge*

with the property that $Z\colon K(\mathcal{A}) \to \mathbb{C}, A \mapsto v(A).z$ satisfies

$$Z(A) \in \mathbb{H} \cup \mathbb{R}_{<0} \quad \text{for all} \quad A \in \mathcal{A},\, A \neq 0.$$

We require moreover local-finiteness and the existence of Harder–Narasimhan filtrations.

In the usual definition, the datum of the heart is replaced by a collection of subcategories $\mathcal{P}(\phi) \subset \mathcal{A}, \phi \in \mathbb{R}$, called *slicing*. The equivalence to the above definition was shown in [Bri07, Prop. 5.3].

The main result about the stability manifold of a K3 surface is the following theorem.

Theorem 4.1. *[Bri07, Cor. 1.3], [Bri08, Thm. 1.1] The set of all stability conditions on a K3 surface X has the structure of a (finite-dimensional) complex manifold $Stab(X)$.*

There is a distinguished connected component $Stab^{\dagger}(X)$ of $Stab(X)$ such that the map $\sigma = (\mathcal{A}, z) \mapsto z$ induces a Galois cover

$$\pi : Stab^{\dagger}(X) \longrightarrow \mathcal{P}_0^+(X).$$

Moreover, the Galois group is identified with the $Aut_0^{\dagger}(\mathcal{D}^b(X)) \subset Aut(\mathcal{D}^b(X))$, the group of auto-equivalences that respect the component $Stab^{\dagger}(X)$ and act trivially on the cohomology $\tilde{H}(X, \mathbb{Z})$.

4.2 Group actions

Given a derived equivalence $\Phi : \mathcal{D}^b(X) \xrightarrow{\sim} \mathcal{D}^b(Y)$ and a stability condition $(\mathcal{A}, z)$ on $\mathcal{D}^b(X)$ we get an induced stability condition $\Phi_*(\mathcal{A}, z) = (\Phi(\mathcal{A}), \Phi^H(z))$ on $\mathcal{D}^b(Y)$. In this way we obtain a left action of the group $Aut(\mathcal{D}^b(X))$ on $Stab(X)$.

There is also a right action of the group

$$\widetilde{Gl}_2^+(\mathbb{R}) = \left\{ (T, f) \mid T \in Gl_2^+(\mathbb{R}),\ f : \mathbb{R} \to \mathbb{R}, f(\phi + 1) = f(\phi) + 1 \text{ with } (\#) \right\}$$

on $Stab(X)$ (cf. [Bri07, Lem. 8.2.]). Here $(\#)$ stands for the condition $\mathbb{R}_{>0}\, T \cdot exp(i\pi\phi) = \mathbb{R}_{>0}\, exp(i\pi f(\phi))$. for all $\phi \in \mathbb{R}$, and $Gl_2^+(\mathbb{R})$ acts on $\mathbb{C}$ via the identification $\mathbb{C} = \mathbb{R}^2$. Note that the projection of the first factor $\widetilde{Gl}_2^+(\mathbb{R}) \to Gl_2^+(\mathbb{R})$ makes $\widetilde{Gl}_2^+(\mathbb{R})$ the universal cover of $Gl_2^+(\mathbb{R})$.

The action admits an easy definition if we use the equivalent description of a stability condition as a pair $(\mathcal{P}, Z)$ consisting of a slicing $\mathcal{P} = \{\mathcal{P}(t) \subset \mathcal{D}^b(X) \,|\, t \in \mathbb{R}\}$ and a stability function $Z : N(X) \to \mathbb{C}$ (c.f. [Bri07, Def. 1.1, Prop. 5.3.]).

Definition 4.2. [Bri07, Lem. 8.2] For all $\sigma = (\mathcal{P}, Z) \in Stab(X)$ and $g = (T, f) \in \widetilde{Gl}_2^+(\mathbb{R})$ we set

$$\sigma \cdot g = (\mathcal{P}', T^{-1} \circ Z).$$

where $\mathcal{P}'(t) = \mathcal{P}(f(\phi))$.

Remark 4.3. The action of $g \in Gl_2^+(\mathbb{R})$ on $Z \in Hom_{\mathbb{Z}}(N(X), \mathbb{C})$ translates to the following action on $N_{\mathbb{C}}$.

$$z \cdot g = (1, i).g^{-1}.\begin{pmatrix} Rez \\ Imz \end{pmatrix} = \Delta^{-1}((dRez - bImz) + i(aImz - cRez))$$

Here $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\Delta = det(g)$. Hence $Gl_2^+(\mathbb{R})$ acts transitively on all oriented bases of the positive plane $\mathbb{R}\langle Rez, Imz\rangle \subset N(X)_{\mathbb{R}}$.

Example 4.4. We can realize arbitrary translations of the slicing via the embedding

$$\Sigma : \mathbb{R} \longrightarrow \widetilde{Gl}_2^+(\mathbb{R}), \quad \lambda \mapsto \Sigma_\lambda = (\exp(i\pi\lambda), \phi \mapsto \phi + \lambda).$$

The image of Σ in $Gl_2^+(\mathbb{R})$ is the unitary group $U(1) \subset GL_1(\mathbb{C}) \subset Gl_2^+(\mathbb{R})$.

The action of the shift $[1]$ equals the action of the translation $\Sigma_1 = (-id, f : \phi \mapsto \phi + 1)$ on the stability manifold.

4.3 Construction of stability conditions

Explicit examples of stability conditions on a K3 surface are constructed as follows.

Fix classes $\beta \in NS(X)_{\mathbb{R}}$ and $\omega \in Amp(X)$ and define a central charge

$$Exp(\beta + i\omega) = (1, \beta + i\omega, \frac{1}{2}(\beta + i\omega)^2) \in N(X)_{\mathbb{C}}.$$

For a torsion free sheaf A of positive rank denote by

$$\mu_\omega^{min}(A) = \inf \{\mu_\omega(Q) \mid A \twoheadrightarrow Q,\, Q \text{ torsion free}\}$$
$$\mu_\omega^{max}(A) = \sup \{\mu_\omega(S) \mid S \hookrightarrow A\}$$

the extremal slopes. Note that $\mu_\omega^{min}(A) \leq \mu_\omega(A) \leq \mu_\omega^{max}(A)$ and that equality holds if and only if A is μ_ω-semi-stable. Define full subcategories of $Coh(X)$ by

$$\mathcal{T} = \{A \in Coh(X) \mid A \,\text{torsion or}\, \mu_\omega^{min}(A/A_{tors}) > \beta.\omega\}$$
$$\mathcal{F} = \{A \in Coh(X) \mid A \,\text{torsion free and}\;\, \mu_\omega^{max}(A) \leq \beta.\omega\}.$$

The following full subcategory of $\mathcal{D}^b(X)$ is a heart of a bounded t-structure.

$$\mathcal{A}(\beta,\omega) = \{E \in \mathcal{D}^b(X) \mid H^0(E) \in \mathcal{T},\, H^{-1}(E) \in \mathcal{F},\, H^i(E) = 0 \text{ if } i \neq 0, -1\}.$$

Theorem 4.5. *[Bri08, Lem. 6.2, Prop. 11.2] The pair*

$$\sigma(\beta,\omega) = (\mathcal{A}(\beta,\omega), z = Exp(\beta + i\omega))$$

is a stability condition on $\mathcal{D}^b(X)$ if following two equivalent properties are satisfied.

1. *For all spherical sheaves A one has $Z(A) \notin \mathbb{R}_{\leq 0}$.*

2. *The vector $\theta(z)$ lies in $\mathcal{L}(X) \subset \mathfrak{D}(X)$.*

The set of all stability conditions arising in this way is denoted by $V(X)$.

Proof. We prove the equivalence of the two properties.

Let z have property (1). We have to show that $v.z \notin \mathbb{R}_{\leq 0}$ for all $v \in \Delta^+(X)$. By a theorem of Yoshioka for all $v \in \Delta^+(X)$ there are μ-stable sheaves A with Mukai vector v. Since $ext^0(A) = ext^2(A) = 1$ by stability and Serre duality, $v(A)^2 = -2$ implies $ext^1(A) = 0$ and hence A is spherical.

For the converse assume that $v.z \notin \mathbb{R}_{\leq 0}$ for all $v \in \Delta^+(X)$. We have to show for all spherical sheaves A we have $Z(A) = Exp(\beta + i\omega).v(A) \notin \mathbb{R}_{\leq 0}$. Note that $v(A) \in \Delta(X)$. If $rk(A) > 0$ then $v(A) \in \Delta^+(X)$ and there is nothing to prove. If $rk(A) = 0$ then $Im(Z(A)) = c_1(A).\omega > 0$ since $\omega \in Amp(X)$ and $c_1(A)$ is an effective class. □

The connected component of $Stab(X)$ containing $V(X)$ is called *distinguished component* and denoted by $Stab^\dagger(X)$.

Let $\Phi : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ be a derived equivalence between two K3 surfaces. We say Φ *respects the distinguished component* if $\Phi_* Stab^\dagger(X) = Stab^\dagger(Y)$.

Remark 4.6. For an object A with Mukai vector $v(A) = (r, l, s)$ with $r \neq 0$ we have explicitly

$$Z(A) = Exp(\beta + i\omega).v(A) = \frac{1}{2r}(v(A)^2 - (l - r\beta)^2 + r^2\omega^2) + i(l - r\beta).\omega,$$

on the other hand if $r = 0$, then $Z(A) = (l.\beta - s) + i(l.\omega)$.

For a sheaf A of positive rank, the condition $Im(Z(A)) = 0$ is equivalent to $\mu_\omega(A) = \beta.\omega$ and in this case the Hodge index theorem implies that $(l - r\beta)^2 \leq 0$ with equality only if $l = r\beta$.

Remark 4.7. The heart $\mathcal{A}(\beta, \omega\lambda)$ is independent of $\lambda > 0$. Indeed, we have $\mu_{\lambda\omega}(A) = \lambda\mu_\omega$ and hence the conditions

$$\mu_\omega^{min}(A/A_{tors}) > \beta.\omega \quad \text{and} \quad \mu_\omega^{max}(A/A_{tors}) \leq \beta.\omega$$

are invariant under $\omega \mapsto \lambda\omega$. Therefore $\mathcal{T}$ and $\mathcal{F}$ do not depend on λ.

Remark 4.8. By [Bri08, Prop. 10.3] the action of $\widetilde{Gl}_2^+(\mathbb{R})$-action on $V(X)$ is free. We introduce the notation $U(X) := V(X) \cdot \widetilde{Gl}_2^+(\mathbb{R}) \cong V(X) \times \widetilde{Gl}_2^+(\mathbb{R})$ for the image.

The following proposition gives an important characterization of $U(X)$.

Proposition 4.9. *[Bri08, Def. 10.2, Prop. 10.3] Let $\sigma = (\mathcal{A}, z)$ be a stability condition on $\mathcal{D}^b(X)$. Then $\sigma \in U(X)$ if and only if the following properties hold.*

1. *All skyscraper sheaves $\mathcal{O}_x$ are stable of the same phase.*
2. *The vector z lies in $\mathcal{P}_0(X)$.*

4.4 Geometric refinements

Remark 4.10. Let us summarize the above discussion in the following diagram:

$$\begin{tikzcd}
U(X) \arrow[r, hook, "\text{open}"'] \arrow[dd] & Stab^\dagger(X) \arrow[rr, "\pi"] \arrow[d, "\widetilde{Gl}_2^+(\mathbb{R})"'] & & \mathcal{P}_0^+(X) \arrow[d, "Gl_2^+(\mathbb{R})"', "\theta"] & \\
& Stab^\dagger(X)/\widetilde{Gl}_2^+(\mathbb{R}) \arrow[rr, "\bar{\pi}"] & & \mathfrak{D}_0^+(X) & \\
V(X) \arrow[ur, hook, "\text{open}"'] \arrow[rrrr, leftrightarrow, "\cong"] & & & & \mathcal{L}(X) \arrow[ul, hook', "\text{open}"]
\end{tikzcd}$$

Here we identify $V(X)$ with its image in $Stab^\dagger(X)/\widetilde{Gl}_2^+(\mathbb{R})$. The maps $\pi, \bar{\pi}$ are covering spaces. Moreover, the map $\pi : U(X) \to \theta^{-1}(\mathcal{L}(X))$ is a covering space with fiber $\mathbb{Z}$.

Lemma 4.11. *Consider the map $\sigma : \mathcal{L}(X) \to V(X)$ which maps $exp(\beta + i\omega)$ to the stability condition $\sigma(\beta, \omega)$.*

Let $\overline{\mathcal{L}}_{0,>2}(X)$ be the closure of $\mathcal{L}_{0,>2} = \mathcal{L}(X) \cap \mathfrak{D}_{0,>2}^+(X)$ in $\mathfrak{D}_{0,>2}^+(X)$, and $\overline{V}_{>2}(X)$ be the intersection of $\overline{V}(X)$ with $\pi^{-1}(\mathfrak{D}_{>2}(X))$.

Then there is a unique continuous extension of $\sigma|_{\mathcal{L}_{0,>2}}(X)$ to an isomorphism

$$\sigma^* : \overline{\mathcal{L}}_{0,>2}(X) \xrightarrow{\sim} \overline{V}_{>2}(X).$$

Proof. Under the isomorphism $exp : T(N(X), v_0) \to \mathfrak{D}(X)$ the set $\overline{\mathcal{L}}_{0,>2}(X)$ gets identified with $\left\{ x + iy \mid y \in \overline{Amp}(X),\ y^2 > 2,\ (*) \right\}$ where $(*)$ is the condition

$$(x + iy).l \notin \mathbb{Z} \quad \text{for all} \quad l \in \Delta(NS(X))$$

As we have $y.l \geq 0$ for $y \in \overline{Amp}(X)$, we can retract $\overline{\mathcal{L}}_{0,>2}(X)$ into the subset $\mathcal{L}(X)_{>2} = \mathcal{L}(X) \cap \mathfrak{D}_{>2}$ via the homotopy $(x + iy, t) \mapsto x + iy + ti\omega$ for $t \in [0, 1]$ and $\omega \in Amp(X)$. It follows that $\overline{\mathcal{L}}_{0,>2}(X)$ is contractible.

Recall from Remark 2.6, that $q = Exp \circ exp^{-1} : \mathfrak{D}(X) \to \mathcal{P}(X)$ is a section of the $Gl_2^+(\mathbb{R})$-bundle $\theta : \mathcal{P}(X) \to \mathfrak{D}(X)$ and that for $exp(x+iy) \in \mathcal{L}(X)$ we have

$$\pi(\sigma(x,y)) = Exp(x+iy) = q(exp(x+iy)).$$

In other words, the following diagram is commutative

$$\begin{tikzcd}
V(X) \arrow[rrr, hook, "\cong"] \arrow[dr] & & & q(\mathcal{L}(X)) \arrow[dl] \arrow[dd, leftrightarrow, "\cong"] \\
& Stab^\dagger(X) \arrow[r, "\pi"] & \mathcal{P}_0^+(X) \arrow[d, "\theta"] & \\
& & \mathfrak{D}_0^+(X) \arrow[u, bend left, "q"] & \mathcal{L}(X). \arrow[l, hook']
\end{tikzcd}$$

As $\overline{\mathcal{L}}_{0,>2}(X)$ is contractible, the restriction of $\pi : Stab^\dagger(X) \to \mathcal{P}_0^+(X)$ to $q(\overline{\mathcal{L}}_{0,>2}(X)) \subset \mathcal{P}_0^+(X)$ is a trivial covering space (cf. Remark 2.6 for the definition of $q = q_{v_0}$) and hence there is a unique section s extending $\sigma \circ \theta : q(\mathcal{L}(X)) \to V(X)$ to $q(\overline{\mathcal{L}}_{0,>2}(X))$. The extension of σ itself is given by

$$s \circ q : \overline{\mathcal{L}}_{0,>2}(X) \longrightarrow Stab^\dagger(X). \qquad \square$$

The set of stability conditions where a given object E is stable is an open subset which is bounded by real, codimension-one submanifolds called walls. This result holds more generally for collections of objects which have a "bounded mass". We recall the precise statements.

Definition 4.12. [Bri08, Def. 9.1] Let $\sigma \in Stab(X)$ be a stability condition and $E \in \mathcal{D}^b(X)$ an object. We define the *mass* of E to be $m_\sigma(E) = \sum_i |Z(A_i)|$ where A_i are the semi-stable factors of E.

A set of objects $S \subset \mathcal{D}^b(X)$ has *bounded mass* if there is a $\sigma \in Stab^\dagger(X)$ with

$$sup\{m_\sigma(E) \,|\, E \in S\} < \infty.$$

In this case also $\{m_{\sigma'}(E) \,|\, E \in S\}$ is bounded for all $\sigma' \in Stab^\dagger(X)$.

Theorem 4.13. *[Bri08, Prop. 9.3] Assume $S \subset \mathcal{D}^b(X)$ has bounded mass and $B \subset Stab^\dagger(X)$ is a compact subset.*
Then there exists a finite collection $W_\gamma, \gamma \in \Gamma$ of real codimension one submanifolds[1] *of $Stab^\dagger(X)$ such that every connected component*

$$C \subset B \setminus \bigcup_{\gamma \in \Gamma} W_\gamma$$

has the following property. If $E \in S$ is semi-stable for one $\sigma \in C$, then E is semi-stable for all $\sigma \in C$. If moreover $E \in S$ has primitive Mukai vector, then E is stable for all $\sigma \in C$.

Lemma 4.14. *Let $0 \neq v \in N(X)$ be a Mukai vector and $B \subset Stab^\dagger(X)$ be a compact subset. Then*

$$S = \{E \in \mathcal{D}^b(X) \,|\, E \ \sigma\text{-semi-stable for some } \sigma \in B, v(E) = v\}$$

has bounded mass.

[1]For us a submanifold is an open subset of a closed submanifold.

Proof. We use the metric d on $Stab(X)$ constructed in [Bri07, Sec. 8]. This metric has the property that

$$|log(m_{\sigma_1}(E)/m_{\sigma_2}(E))| \leq d(\sigma_1, \sigma_2)$$

for all $0 \neq E \in \mathcal{D}^b(X)$. As $d(\sigma_1, \sigma_2) < \infty$ for $\sigma_1, \sigma_2 \in Stab^\dagger(X)$ and B is compact the following constants are finite.

$$M = \max\{d(\sigma, \sigma') \,|\, \sigma, \sigma' \in B\}, \quad N = \max\{|Z_\sigma(v)| \,|\, \sigma \in B\}$$

It follows that $m_\tau(E), E \in S$ is bounded by $exp(M)N$. □

Next, we recall Bridgeland's description of the boundary of $U(X)$. As a well known consequence we get a covering of the stability manifold by certain translates of the closure $\overline{U}(X)$.

Before we state Bridgeland's theorem, recall that an object $A \in \mathcal{D}^b(X)$ is called *spherical* if $Ext^k(A, A) = \mathbb{C}$ for $k = 0, 2$ and zero otherwise. Examples of spherical objects are line bundles and $\mathcal{O}_C$ for a (-2)-curve $C \subset X$. To a spherical object we can associate an auto-equivalence of the derived category.

Theorem 4.15 (Seidel–Thomas)**.** *[Huy06, Sec. 8.1.] Let $A \in \mathcal{D}^b(X)$ be a spherical object. There exists an auto-equivalence $T_A : \mathcal{D}^b(X) \to \mathcal{D}^b(X)$ and for each object E an exact triangle*

$$\longrightarrow \oplus_i Hom(A, E[i]) \otimes A[-i] \longrightarrow E \longrightarrow T_A E \xrightarrow{+1} .$$

Theorem 4.16. *[Bri08, Thm. 12.1.] The boundary $\partial U(X)$ is contained in a locally finite union of real codimension-one submanifolds. If $x \in \partial U(X)$ is a general boundary point, i.e. lies only on one of these submanifolds, then precisely one of the following possibilities hold.*

(A^+) *There is a rank r spherical vector bundle A such that the stable factors of the objects $\mathcal{O}_x, x \in X$ are A and $T_A(\mathcal{O}_x)$.*

(A^-) *There is a rank r spherical vector bundle A such that the stable factors of the objects $\mathcal{O}_x, x \in X$ are $A[2]$ and $T_A^{-1}(\mathcal{O}_x)$.*

(C_k) *There is a non-singular rational curve C and an integer k such that $\mathcal{O}_x$ is stable if and only if $x \notin C$. If $x \in C$, then it has a stable factor $\mathcal{O}_C(k)[1]$.*

Remark 4.17. If $\sigma \in \overline{U}(X)$ satisfies condition (A^+) or (A^-), then the central-charges of $\mathcal{O}_x$ and A are co-linear: $Z(A)/Z(v_0) \in \mathbb{R}_{>0}$. This is precisely the condition we used in Definition 2.9 to define the A-type wall $W_A(v(A), v_0) \subset \mathfrak{D}(X)$. Therefore the image $\overline{\pi}(\sigma)$ of in $\mathfrak{D}(X)$ lies on $W_A(v(A), v_0)$.

Similarly, if $\sigma \in \overline{U}(X)$ satisfies condition (C_k), then $\overline{\pi}(\sigma)$ lies on the wall $W_C(v(\mathcal{O}_C), v_0)$ of type C.

Definition 4.18. Let $\widetilde{W}(X) \subset Aut(\mathcal{D}^b(X))$ be the group generated by the spherical twists $T_A^2, T_{\mathcal{O}_C(k)}$ for all (-2)-curves C, $k \in \mathbb{Z}$ and spherical vector bundles A, which occur in the description of the boundary $\partial U(X)$ given in Theorem 4.16.

Remark 4.19. One can check, that all equivalences $\Phi \in \widetilde{W}(X)$ have the property $\Phi^H(v_0) = v_0$. This means we get a map

$$\widetilde{W}(X) \longrightarrow W^0(N(X), v_0) \subset O(N(X)), \quad \Phi \mapsto \Phi^H.$$

As we will see in Proposition 5.8, this map is surjective.

Remark 4.20. All equivalences $\Phi \in \widetilde{W}(X)$ respect the distinguished component $Stab^\dagger(X)$. Indeed, the spherical twists $T_{\mathcal{O}_C(k)}$ and T_A^2, which generate $\widetilde{W}(X)$, map the corresponding boundary components of $\partial U(X)$ into $\partial U(X)$, cf. [Bri08, Thm. 12.1.]. We will study equivalences with this property more closely in chapter 5.

The following lemma is an easy consequence of the proof of [Bri08, Prop. 13.2.].

Lemma 4.21. *The translates of the closed subset $\overline{U}(X)$ under the group $\widetilde{W}(X)$ cover $Stab^\dagger(X)$:*

$$\bigcup_{\Phi \in \widetilde{W}(X)} \Phi_* \overline{U}(X) = Stab^\dagger(X).$$

One can show, moreover, that the intersections of the interiors $U(X) \cap \Phi_* U(X)$ are empty unless $\Phi = id$. However, we will not need this refinement.

Chapter 5

Equivalences respecting $Stab^\dagger(X)$

Let $\Phi : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ be a derived equivalence between two K3 surfaces. Recall from chapter 4, that Φ *respects the distinguished component* if $\Phi_* Stab^\dagger(X) = Stab^\dagger(Y)$.

As we will see, this property can be verified for most of the known equivalences. It is expected that $Stab(X)$ is connected and therefore it should in fact hold always.

We will use the following criterion, which is an easy consequence of [Bri08, Prop. 10.3] cf. Proposition 4.9.

Corollary 5.1. *Let $\Phi : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ be a derived equivalence between two K3 surfaces. If the objects $\Phi(\mathcal{O}_x), x \in X$ are σ-stable of the same phase for some $\sigma \in Stab^\dagger(Y)$, then Φ preserves the distinguished component.*

Proof. The objects $\Phi(\mathcal{O}_x), x \in X$ are σ-stable of the same phase if and only if the sheaves $\mathcal{O}_x, x \in X$ are $\Phi_*^{-1}(\sigma)$-stable of the same phase. We now apply Proposition 4.9 which tells us that $\Phi_*^{-1}(\sigma) \in U(X)$ and hence Φ preserves the distinguished component. Therefore we need to check that $(\Phi^{-1})^H z \in \mathcal{P}_0(X)$, where z is the central charge of σ. This follows from $z \in \mathcal{P}_0(X)$ and the fact that $\Phi^H : N(X) \to N(Y)$ is an isometry. $\square$

Lemma 5.2. *The equivalences of derived categories listed below respect the distinguished component of the stability manifold.*

- *Shifts:* $[1] : A \mapsto A[1]$
- *Isomorphisms: For $f : X \cong Y$, the functor $f_* : A \mapsto f_* A$*
- *Line bundle twists: For $L \in Pic(X)$, the functor $A \mapsto L \otimes A$*

Proof. Shifts. We have seen in Example 4.4 that $[1]_*\sigma = \sigma \cdot \Sigma_1$. Hence $t \mapsto \sigma \cdot \Sigma_t, t \in [0,1]$ is a continuous path from σ to $[1]_*\sigma$ in $Stab(X)$. If follows that $[1]_*\sigma$ lies in the same connected component as σ.

Isomorphisms. In the skyscraper sheaves $f_*(\mathcal{O}_x) = \mathcal{O}_{f(x)}$ are stable of phase one in any $\sigma \in V(X)$. By Corollary 5.1 the functor f_* preserves the distinguished component.

Line bundle twists. Let $L \in Pic(X)$. It is $L \otimes \mathcal{O}_x \cong \mathcal{O}_x$ for all $x \in X$. Again, these sheaves are stable of phase one in any $\sigma \in V(X)$ and we can apply Corollary 5.1. □

Before we can deal with more interesting auto-equivalences, we need another digression on stability conditions.

5.1 Large volume limit

Following [Bri08, Prop. 14.2] we will show that families of Gieseker-stable sheaves give rise to families of σ-stable objects in stability conditions σ near the large volume limit.

Let M be a quasi-compact scheme over $\mathbb{C}$. Denote by $i_m : X \to X \times M$ the inclusion of the fiber over $m \in M(\mathbb{C})$. For a sheaf $E \in Coh(X \times M)$ denote by E_m the restriction $i_m^* E \in Coh(X)$ to the fiber over m.

Proposition 5.3. *Let $h \in NS(X)$ be an ample class. Let $\mathcal{E} \in Coh(M \times X)$ be an M-flat family of Gieseker-stable sheaves of fixed Mukai vector $v(E_m) = v \in N(X)$. Assume that $r(E_m) > 0$ and $\mu(E_m) = \mu_h(E_m) > 0$.*

Then there exists a $n_0 \geq 1$ such that the objects $E_m, m \in M(\mathbb{C})$ are stable with respect to the stability condition $\sigma(0, nh) \in V(X)$ for all $n \geq n_0$.

Proof. We show that Bridgeland's arguments suffice to cover our situation. We first note that the heart $\mathcal{A}(0, nh)$ is independent of n. Moreover, the objects E_m lie in the heart $\mathcal{A}(0, nh)$ since E_m is stable of $\mu(E_m) > 0$.

In the following we abuse the notation by writing E for E_m when the dependence from $m \in M(\mathbb{C})$ is not essential. Suppose $0 \neq A \to E$ is a proper sub-object of E in $\mathcal{A}(0, h)$. We have to find an $n_0 \geq 1$ independent of A and m, such that $\arg(Z_n(A)) < \arg(Z_n(E))$ for all $n \geq n_0$. We have the following formula

$$\frac{Z_n(E)}{r(E)} - \frac{Z_n(A)}{r(A)} = -(\nu(E) - \nu(A)) + in(\mu(E) - \mu(A)) =: \Delta_n. \tag{5.1}$$

The inequality $\arg(Z_n(A)) < \arg(Z_n(E))$ is equivalent to

$$\arg(Z_n E) < \arg(\Delta_n). \tag{*}$$

The morphism $A \to E$ induces an exact sequence of cohomology sheaves:

$$0 \longrightarrow D \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0 \tag{5.2}$$

where $D = \mathcal{H}^{-1}(Cone(A \to E))$ and $B = \mathcal{H}^0(Cone(A \to E))$. Note that, $A = \mathcal{H}^0(A)$ since $\mathcal{H}^{-1}(A) = \mathcal{H}^{-1}(E) = 0$. Let $E' \subset E$ be the image of $A \to E$. We get short exact sequences

$$0 \longrightarrow D \longrightarrow A \longrightarrow E' \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow E' \longrightarrow E \longrightarrow B \longrightarrow 0. \tag{5.3}$$

Note that A is an extension of torsion free sheaves, and therefore torsion free. By construction of $\mathcal{A}(0, h)$ we have $\mu^{min}(A) > 0$ and $\mu^{max}(D) \leq 0$ unless $D = 0$. If $D = 0$, then $A \to E$ is a subsheaf. The stability of E is equivalent to $\Delta_n \in \mathbb{H} \cup \mathbb{R}_{<0}$. If moreover $\mu(A) = \mu(E)$, i.e. $\Delta_n \in \mathbb{R}_{<0}$, then $(*)$ holds always,

so we can exclude this case henceforth.
If $D \neq 0$, then we have $\mu(D) \leq 0 < \mu(A)$ and therefore also $\mu(A) < \mu(E')$. Hence

$$\mu(A) < \mu(E') \leq \mu(E) \tag{5.4}$$

by stability of E. This shows that $\Delta_n \in \mathbb{H}$.

Note that also $Z_n(E) \in \mathbb{H}$, as can be easily seen from the definition. It follows, that the inequality $(*)$ holds, if and only if

$$\frac{Re(Z_n(E))}{Im(Z_n(E))} > \frac{Re(\Delta_n)}{Im(\Delta_n)} = -\frac{\nu(E)-\nu(A)}{n(\mu(E)-\mu(A))}. \tag{\#}$$

The quotient $Re(Z_n(E))/Im(Z_n(E))$ converges to $+\infty$ for $n \to \infty$. Hence it suffices to bound $-(\nu(E)-\nu(A))/n(\mu(E)-\mu(A))$ from above.

Lemma 5.4. *The set*

$$\{\nu(A) \,|\, A \to E_m \text{ sub-object in } \mathcal{A}(0,h),\ m \in M(\mathbb{C})\}$$

is bounded above.

We postpone the proof. To show that the inequality (#) holds for large n we have to find a constant C such that $\mu(E) - \mu(A) > C > 0$.
Case $\mu(E') < \mu(E)$: As $r(E') \leq r(E)$ we have $\mu(E) - \mu(E') > 1/r(E)^2$ and by (5.4) the same bound holds for $\mu(E) - \mu(A)$ as well.
Case $\mu(E') = \mu(E)$: If $D = 0$, then $A \to E$ is a subsheaf and again $\mu(E) - \mu(A) > 1/r(E)^2$. If $D \neq 0$, then the exact sequence (5.3) and $\mu(D) \leq 0$ shows that

$$\begin{aligned}
\mu(A) &= \mu(D)\frac{r(D)}{r(E')+r(D)} + \mu(E')\frac{r(E')}{r(E')+r(D)} \\
&\leq \mu(E')\frac{r(E')}{r(E')+r(D)} \leq \mu(E)\frac{r(E)}{r(E)+r(D)} \\
&\leq \mu(E)\frac{r(E)}{r(E)+1}.
\end{aligned}$$

As $r(E) = r(E_m)$ is independent of m we get a uniform bound. □

Proof of Lemma. Recall that if $v(A) = (r, l, s)$, then $\nu(A) = s/r$. The Euler-characteristic of A is computed to

$$\chi(A) = \chi(\mathcal{O}_X, A) = h^0(A) - h^1(A) + h^2(A) = -(1,0,1).(r,l,s) = r + s$$

hence it suffices to bound $\chi(A)/r$ from above. As $\mu^{min}(A) > 0 = \mu(\mathcal{O}_X)$ we have $Hom(A, \mathcal{O}_X) = H^2(A) = 0$. Therefore $\chi(A)/r \leq h^0(A)/r$. The long exact sequence

$$0 \longrightarrow H^0(D) \longrightarrow H^0(A) \longrightarrow H^0(E') \longrightarrow H^1(D) \longrightarrow \dots$$

shows that $h^0(A) \leq h^0(D) + h^0(E')$. Moreover, $h^0(E') \leq h^0(E)$ and $h^0(E) = h^0(E_m)$ is bounded uniformly in $m \in M(\mathbb{C})$ since $h^0(E_m)$ is semi-continuous and M quasi-compact. In the case $D = 0$ we are done. Let now $D \neq 0$. Note that $r = r(A) \geq r(D)$, and hence $h^0(D)/r(A) \leq h^0(D)/r(D)$. Therefore it suffices to bound

$$\left\{ h^0(D)/r(D) \mid D = \mathcal{H}^{-1}(Cone(A \to E_m)), m \in M(\mathbb{C}) \right\}.$$

We claim that $h^0(D)/r(D) \leq 1$ for all $D \neq 0$ torsion free with $\mu^{min}(D) \leq 0$.
Let $0 \neq s \in H^0(D)$ be a section. We claim that $K = ker(s : \mathcal{O}_X \to D) = 0$: If not, then $\mu(K) < \mu(\mathcal{O}_X)$ by μ-stability of $\mathcal{O}_X$. Therefore $0 = \mu(\mathcal{O}_X) < \mu(Im(s))$, which contradicts $\mu^{min}(D) \leq 0$.
We get an exact sequence:

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{s} D \longrightarrow Q \longrightarrow 0. \tag{5.5}$$

We claim that Q is torsion free: Let $T \subset Q$ be the torsion subsheaf and S be the kernel of the map $D \to Q \to T/Q$. There is an induced exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow S \longrightarrow T \longrightarrow 0. \tag{5.6}$$

If $dim(T) = 1$, then $\mu(S) = c_1(T).h > 0$. This contradicts the stability of D as $S \subset D$ and $\mu^{min}(D) \leq 0$.
If $dim(T) = 0$, then $Ext^1(T, \mathcal{O}_X) = H^1(T)^\vee = 0$ and therefore the exact sequence (5.6) splits. This contradicts D being torsion free.
We conclude that Q is torsion free.

As Q is a quotient of D we find that $\mu^{min}(Q) \geq \mu^{min}(D) \geq 0$ unless $Q = 0$. This shows that Q fulfills the same assumptions as D in our claim. We can therefore apply induction on $r(D)$.
Case $r(D) = 1$: Then $H^0(D) = 0$ as D has no non-trivial torsion-free quotients.
Case $r(D) > 1$. We may assume that $h^0(Q)/r(Q) \leq 1$ by induction. The long exact sequence associated to (5.5) shows that

$$\frac{h^0(D)}{r(D)} = \frac{h^0(Q) + 1}{r(Q) + 1} \leq 1$$

and therefore completes the proof. □

5.2 Moduli spaces and spherical twists

A very important class of derived equivalences between K3 surfaces is provided by moduli spaces of sheaves.

Theorem 5.5 (Mukai, et. al.)**.** *[Huy06, Sec. 10.2] Fix a standard vector $v = (r, l, s) \in N(X)$ with $r > 0$.*
There is an ample class $h \in NS(X)$ such that the moduli space $M = M_h(v)$ of Gieseker-semi-stable sheaves of Mukai vector v has the following properties.

1. *There is a universal family $\mathcal{E}$ on $M \times X$, i.e. M is a fine moduli space.*
2. *All parametrized sheaves are actually stable.*
3. *M is a K3 surface, in particular smooth, projective and non-empty.*
4. *The Hodge structure $H^2(M, \mathbb{Z})$ is canonically identified with the subquotient $v^\perp/v$ of $\tilde{H}(X, \mathbb{Z})$.*

5. *The universal family induces a derived equivalence*

$$FM(\mathcal{E}) : \mathcal{D}^b(M) \longrightarrow \mathcal{D}^b(X),\ A \mapsto \mathbb{R}pr_{2*}(\mathrm{pr}_1^* A \otimes^{\mathbb{L}} \mathcal{E}).$$

The next proposition shows, that all derived equivalences of this form respect the distinguished component.

Proposition 5.6. *Let $M = M_h(v)$ be a fine, compact, two-dimensional moduli-space of Gieseker-stable sheaves on X and $\Phi : \mathcal{D}^b(M) \xrightarrow{\sim} \mathcal{D}^b(X)$ the Fourier–Mukai equivalence induced by the universal family (cf. [Huy06, Sec. 10.2]). Then Φ respects the distinguished component.*

Proof. We first reduce to the case $\mu(E) > 0$:

Let $\mathcal{O}(1)$ be the ample line bundle with $c_1(\mathcal{O}(1)) = h$. Write $v = (r, l, s)$ and fix a number $n_0 \geq 0$ such that $(l + n_0 rh).h > 0$. Then the sheaves $E(n_0) = E \otimes \mathcal{O}(1)^{\otimes n_0}, [E] \in M$ have slope $\mu(E(n_0)) > 0$.

Note that $E(n_0)$ is Gieseker-stable since E is Gieseker-stable. Indeed, the Hilbert polynomials are related by $\chi(E(n_0)(m)) = \chi(E(n_0 + m))$. Moreover, $S \mapsto S(n_0)$ induces a bijection between the set of proper subsheaves of E and $E(n_0)$. Finally we have the trivial relation

$$\frac{\chi(S(m))}{r(S)} < \frac{\chi(E(m))}{r(E)} \iff \frac{\chi(S(n_0 + m))}{r(S)} < \frac{\chi(E(n_0 + m))}{r(E)}.$$

for all large $m \gg 0$.

Let $\mathcal{E} \in Coh(M \times X)$ be the universal family. Then the sheaf $\mathcal{E}' = \mathcal{E} \otimes \mathrm{pr}_2^*(\mathcal{O}(n))$ makes M the moduli space of Gieseker-stable sheaves of Mukai vector $v' = exp(nh).v$. By Lemma 5.2 the equivalence $FM(\mathcal{E})$ respects the distinguished component if and only if $FM(\mathcal{E}') = (\mathcal{O}(n) \otimes \_) \circ FM(\mathcal{E})$ does.

Therefore we may assume, without loss of generality, that $\mu(E) > 0$ for all $[E] \in M$. Hence the universal family $\mathcal{E} \in Coh(M \times X)$ fulfills the assumptions of Proposition 5.3. It follows, that there is an $n \geq 0$ such that all the sheaves $E_m, m \in M(\mathbb{C})$ are stable in the stability condition $\sigma(0, nh) \in U(X)$.

As the sheaves $E_m, m \in M$ are Gieseker-stable of positive slope $\mu(E_m) > 0$, they lie in the heart $\mathcal{A}(0, nh) = \mathcal{A}(0, h)$ and therefore have the same phase. Finally note that $E_m = FM(\mathcal{E})(\mathcal{O}_m)$. Hence the proposition follows from Corollary 5.1. □

Proposition 5.7. *Let A be a spherical vector bundle, which is Gieseker-stable with respect to an ample class $h \in NS(X)$.*
Then the spherical twist $T_A : \mathcal{D}^b(X) \to \mathcal{D}^b(X)$ respects the distinguished component.

Proof. The spherical twist functor has Fourier–Mukai kernel

$$\mathcal{P} = Cone(pr_1^* A^\vee \otimes pr_2^* A \xrightarrow{tr} \mathcal{O}_\Delta) \in \mathcal{D}^b(X \times X)$$

cf. [Huy06, Def. 8.3], i.e. $T_A = FM(\mathcal{P})$.
Claim: The complex $\mathcal{E} = \mathcal{P}[-1]$ is quasi-isomorphic to a sheaf which is flat along pr_1.
Let $i_x : X \to \{x\} \times X \subset X \times X$ be the inclusion of the fiber. By [Huy06, Lem.

3.31.] it suffices to show that $\mathbb{L}i_x^*\mathcal{E} \in \mathcal{D}^b(X)$ is a sheaf for all $x \in X$. Now $\mathbb{L}i_x^*\mathcal{P} = Cone(tr_x)$, where

$$tr_x : A_x{}^\vee \otimes A \longrightarrow A_x{}^\vee \otimes A_x \xrightarrow{tr} \mathcal{O}_x$$

and $A_x = A \otimes \mathcal{O}_x$ is the fiber of A at $x \in X$. As tr_x is clearly surjective, we find $\mathbb{L}i_x^*\mathcal{P} = ker(tr_x)[1]$. Hence $\mathbb{L}i_x^*\mathcal{E} = ker(tr_x)$ is a sheaf.

Mukai shows in [Muk87, Rem. 3.11.] that the sheaves $T_A(\mathcal{O}_x) = ker(tr_x) = E_x$ are Gieseker-stable. Therefore $\mathcal{E}$ induces a map $f : X \to M_h(v_1)$, where $v_1 = v(T_A(\mathcal{O}_x))$. As $v_1^2 = 0$ the moduli space $M = M_h(v_1)$ is a K3 surface.

Claim: f is injective on $\mathbb{C}$-points. Indeed, let $f(x) = [E_x]$ be a point in the image of f and set $F = E_x{}^{\vee\vee}$. We get an exact sequence

$$0 \longrightarrow E_x \longrightarrow F \longrightarrow Q \longrightarrow 0.$$

By [Muk87, Prop. 3.9.] the quotient Q is a isomorphic to $\mathcal{O}_x$. This shows the claim.

Claim: $d_x f$ is an isomorphism. Indeed, we the differential $d_x f$ is the composition

$$T_x X = Ext^1_X(\mathcal{O}_x, \mathcal{O}_x) \xrightarrow{T_A} Ext^1_M(E_x, E_x) = T_{f(x)} M$$

which is an isomorphism as T_A is an equivalence.

It follows that f is an isomorphism. In other words $\mathcal{E}$ makes X the moduli space of Gieseker-stable sheaves $M_h(v_1)$. It follows from Proposition 5.6 that $FM(\mathcal{E}) = T_A$ respects the distinguished component. □

Using Bridgeland's description the boundary of $U(X)$ in [Bri08, Thm. 12.1] (cf. Theorem 4.16) we can also show that spherical twists along torsion sheaves do respect the distinguished component.

Proposition 5.8. *Let C be a (-2)-curve on a K3 surface X and $k \in \mathbb{Z}$, then the spherical twist $T_{\mathcal{O}_C(k)}$ does respect the distinguished component.*

Proof. We will show that every pair (C, k) does define a non-empty boundary component of $U(X)$ of type (C_k). Then [Bri08, Thm. 12.1] shows that $T_{\mathcal{O}_C(k)}{}_* \overline{U}(X) \cap \overline{U}(X) \neq \emptyset$ and therefore $T_{\mathcal{O}_C(k)}$ respects the distinguished component, cf. Remark 4.20.

We first show, that every (-2)-curve defines a boundary component of the ample cone.

Lemma 5.9. *There is a class $\eta \in \overline{Amp(X)}$ such that $C.\eta = 0$ and $C'.\eta \geq 1$ for all other (-2)-curves C'.*

Proof of Lemma. Choose an integral ample class $h \in Amp(X)$ and consider

$$\eta(t) = h + t(C.h)C \ \in NS(X)_\mathbb{R}, \quad t > 0.$$

We have $\eta(t).C' \geq 1$ since $(h.C') \geq 1$ and $(C.C') \geq 0$ for all (-2)-curves $C' \neq C$. Moreover, $\eta(0).C > 0$ and $\eta(t).C < 0$ for $t \gg 0$ since $C.C = -2$. Thus there is a (unique) $t_0 > 0$ with $\eta(t_0).C = 0$. Setting $\eta = \eta(t_0)$ proves the Lemma. □

Multiplying with a positive number we can assume that $\eta^2 > 2$. We claim that there is always a $\beta \in NS(X)_\mathbb{R}$ such that the following conditions are fulfilled:

1. $exp(i\eta + \beta).\delta \neq 0$ for all $\delta \in \Delta(X)$, i.e. $exp(i\eta + \beta) \in \mathcal{P}_0^+(X)$.
2. $exp(i\eta + \beta).\delta \notin \mathbb{R}_{\leq 0}$ for all $\delta \in \Delta^{>0}(X)$ and
3. $\beta.C + k \in (-1, 0)$.

Indeed, for $\delta = (r, l, s)$ we have

$$Im(exp(i\eta + \beta).\delta) = l.\eta - r\beta.\eta.$$

This number is non-zero if $r \neq 0$ and $\beta.\eta \neq l.\eta/r$. If $r = 0$, then $\delta^2 = l^2 = -2$ and $exp(i\eta+\beta).\delta = l.\eta = 0$ implies that $l = \pm C$ by construction of η. In this case $Re(exp(i\eta+\beta).\delta) = \pm\beta.C - s$ is nonzero if (3) is fulfilled. Thus it suffices to chose β in such a way that the countably many inequalities $\beta.\eta \neq l.\eta/r$, $l \in NS(X)$ and the open condition (3) hold. This shows the claim.

Choose an ample class ω with $\omega^2 > 2$. Let $\omega_t, 0 \leq t \leq 1$ be a path from ω to η such that $\omega_t^2 > 2$ and $\omega_t \in Amp(X)$ for $0 \leq t < 1$. This ensures that the family of central charges $exp(i\omega_t + \beta), 0 \leq t < 1$ lies in $\mathcal{L}(X)$ (see Lemma 2.11) and hence there is a unique stability condition $\sigma(t) = \sigma_X(\beta, \omega_t) \in V(X), 0 \leq t < 1$ with central charge $exp(i\omega_t + \beta)$.

By the covering space property there is a unique limiting stability condition $\sigma(1)$ with central charge $exp(i\eta + \beta)$.

Note that $\sigma(1)$ does not lie on a boundary component of type $A^{\pm}$ by (2). If $x \in C$ then $\mathcal{O}_x$ is destabilized in $\sigma(1)$ by a sequence

$$0 \longrightarrow \mathcal{O}_C(n+1) \longrightarrow \mathcal{O}_x \longrightarrow \mathcal{O}_C(n)[1] \longrightarrow 0.$$

Thus $\sigma(1)$ is a point of a boundary component of $U(X)$ of type (C_n) for some n. By Lemma 5.9 the stability condition $\sigma(1)$ does not lie on any other wall of type (C).

The number n is uniquely determined by the property that

$$Z_{\sigma(1)}(\mathcal{O}_C(n)[1]),\ Z_{\sigma(1)}(\mathcal{O}_C(n+1)) \in \mathbb{H} \cup \mathbb{R}_{<0}$$

which has to hold since $\mathcal{O}_C(n+1)$ and $\mathcal{O}_C(n)[1]$ lie in the heart of $\sigma(1)$. This is equivalent to $-1 \leq \beta.C + n < 0$. Hence $k = n$ by condition (3). □

Lemma 5.10. *Let A be a spherical vector bundle defining a boundary component of $U(X)$, Then T_A respects the distinguished component.*

Note that we do not require A to be Gieseker-stable and hence Proposition 5.7 does not apply. Moreover, Remark 4.20 only gives the weaker statement, that T_A^2 respects the distinguished component.

Proof. Bridgeland proves in [Bri08, Lem. 12.1] that for a general σ on the wall (A^+) the sequence

$$0 \longrightarrow A^{\oplus r} \longrightarrow \mathcal{O}_x \longrightarrow T_A(\mathcal{O}_x) \longrightarrow 0$$

is a Jordan-Hölder filtration of $\mathcal{O}_x$. In particular, $T_A\mathcal{O}_x$ is σ-stable and has the same phase as $\mathcal{O}_x$, namely 1. Thus we can apply Corollary 5.1 again. □

Remark 5.11. The general question, if for a spherical object $A \in \mathcal{D}^b(X)$ the equivalence T_A respects the distinguished component remains open – even in the case that A is a vector bundle.

5.3 Auto-equivalences and the Kähler moduli space

It was shown by [HLOY04],[Plo05] and [HMS09] (cf. Theorem 3.1) that the image of the map

$$Aut(\mathcal{D}^b(X)) \longrightarrow O_{Hodge}(\tilde{H}(X,\mathbb{Z}))$$

is the index two subgroup $O^+_{Hodge}(\tilde{H}(X,\mathbb{Z}))$.

Proposition 5.12. *Let $Aut^\dagger(\mathcal{D}^b(X)) \subset Aut(\mathcal{D}^b(X))$ be the subgroup of auto-equivalences which respect the distinguished component. Then*

$$Aut^\dagger(\mathcal{D}^b(X)) \longrightarrow O^+_{Hodge}(\tilde{H}(X,\mathbb{Z}))$$

is surjective.

Proof. As explained in [Huy06, Cor. 10.13.] every element of $O^+_{Hodge}(\tilde{H}(X,\mathbb{Z}))$ is induced by the composition of derived equivalences of the following type.

1. Line bundle twists: For $L \in Pic(X)$, the functor $L \otimes \_ \in Aut(\mathcal{D}^b(X))$.
2. For isomorphisms $f : X \to Y$, the functor $f_* : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$.
3. For fine, compact, two-dimensional moduli spaces M of Gieseker-stable sheaves with universal family $\mathcal{E}$, the Fourier–Mukai transform
$$FM(\mathcal{E}) : \mathcal{D}^b(M) \to \mathcal{D}^b(X).$$
4. Spherical twists along $\mathcal{O}_X$.
5. Spherical twists along $\mathcal{O}_C$ for a (-2)-curve $C \subset X$.

All these equivalences do respect the distinguished component due to our Lemma 5.2 for (1),(2), Proposition 5.6 for (3), Proposition 5.7 for (4) and Proposition 5.8 for (5). □

This result enables us prove the alternative description of the Kähler moduli space using the stability manifold, alluded to in Remark 3.4.

Corollary 5.13. *We have*

$$Aut^\dagger(\mathcal{D}^b(X)) \setminus Stab^\dagger(X)/\widetilde{Gl}_2^+(\mathbb{R}) \cong KM_0(X)$$

where $KM_0(X) = \Gamma_X \setminus \mathfrak{D}_0(N(X)) \subset KM(X)$.

Proof. Recall that $Aut_0^\dagger(\mathcal{D}^b(X))$ is the subgroup of $Aut^\dagger(\mathcal{D}^b(X))$ consisting of auto-equivalences acting trivially on $\tilde{H}(X,\mathbb{Z})$. By [Bri08, Thm. 1.1] (cf. Theorem 4.1) the quotient $Aut_0^\dagger(\mathcal{D}^b(X)) \setminus Stab^\dagger(X)$ is identified with the period domain $\mathcal{P}_0^+(X) \subset N(X)_\mathbb{C}$ via $\pi : Stab^\dagger(X) \to \mathcal{P}_0^+(X)$. As π is $\widetilde{Gl}_2^+(\mathbb{R})$-equivariant, we have

$$Aut_0^\dagger(\mathcal{D}^b(X)) \setminus Stab^\dagger(X)/\widetilde{Gl}_2^+(\mathbb{R}) \cong \mathcal{P}_0^+(X)/Gl_2^+(\mathbb{R}) \cong \mathfrak{D}_0^+(X).$$

Now Proposition 5.12 shows that

$$Aut^\dagger(\mathcal{D}^b(X)) \setminus \mathfrak{D}_0^+(X) \cong O^+_{Hodge}(\tilde{H}(X,\mathbb{Z})) \setminus \mathfrak{D}_0^+(X) \cong KM_0(X). \qquad \square$$

5.4 Reduction to the large volume limit

As another consequence we obtain the following proposition which allows us to reduce many statements about objects with a standard Mukai vector v (cf. Definition 3.7) to the special case $v = v_0 = (0,0,1) \in N(X)$, which is the class of a point sheaf.

Proposition 5.14. *Let $v \in N(X)$ be a standard vector. Then there is a K3 surface Y and a derived equivalence $\Phi : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ such that*

$$\Phi^H(v) = v_0$$

and Φ respects the distinguished component.

Moreover, Y is a fine moduli space of Gieseker-stable sheaves and the Hodge structure $H^2(Y,\mathbb{Z})$ isomorphic to the subquotient $v^\perp/v$ of $\tilde{H}(X,\mathbb{Z})$.

Proof. Write $v = (r,l,s)$. *We may assume that $r > 0$:*
If $r < 0$ then a shift [1] maps v to $-v$ which has $r(-v) > 0$.
If $r = 0$ and $s \neq 0$ then $v' = T^H_{\mathcal{O}_X}(v) = (-s, al, -r)$ has $r(v') \neq 0$, and we are done after applying a shift if necessary.
If $r = 0$ and $s = 0$ then $l \neq 0$ and we find a line bundle L, with $c_1(L) = l' \in NS(X)$ such that $l.l' \neq 0$. A line bundle twist along L maps v to

$$exp(l') \cdot v = (0, l, l.l')$$

and we are in the situation considered in the last step.

Note that the equivalences $\mathcal{L} \otimes \_$ and $T_{\mathcal{O}_X}$ respect the distinguished component by Lemma 5.2 and Proposition 5.7.

By Theorem 5.5 the moduli space of Gieseker-stable sheaves $Y = M_h(v)$ is a K3 surface with Hodge structure $H^2(Y,\mathbb{Z}) \cong v^\perp/v$ as subquotient of $\tilde{H}(X,\mathbb{Z})$. The derived equivalence induced by the universal bundle $\mathcal{E}$

$$FM(\mathcal{E}) : \mathcal{D}^b(M_h(v)) \longrightarrow \mathcal{D}^b(X)$$

respects the distinguished component by Proposition 5.6.

Moreover, as $FM(\mathcal{E})(\mathcal{O}_y) = E_y$ is the vector bundle parametrized by $y \in M_h(v)$ we find that $FM(E)^H(v_0) = v$. $\square$

Chapter 6

Cusps and hearts of stability conditions

In this chapter we prove our main geometric results about cusps and stability conditions. We will introduce the notion of a linear degeneration to a cusp in the Kähler moduli space and classify all paths in the stability manifold mapping to linear degenerations. Moreover, we construct paths in the stability manifold with special limiting hearts.

6.1 Linear degenerations in $\overline{KM}(X)$

Definition 6.1. Let $\gamma(t) \in KM(X), t \gg 0$ be a path in the Kähler moduli space and $[v]$ a standard cusp of $\overline{KM}(X)$.

We say $\gamma(t)$ is a *linear degeneration* to a cusp $[v] \in \overline{KM}(X)$ if there exists a lift $\alpha(t)$ of $\gamma(t)$ to $\mathfrak{D}^+(X)$ and a vector $w \in \Gamma_X \cdot v$ such that

$$\alpha(t) = exp_w(x_0 + i\,t\,y_0)$$

for some $x_0 \in A(w)_{\mathbb{R}},\ y_0 \in C(L(w))$.

Proposition 6.2. *Let $\gamma(t)$ be a linear degeneration to $[v] \in \overline{KM}(X)$.*

1. *The limit of $\gamma(t)$ in $\overline{KM}(X)$ is*

$$\lim_{t\to\infty} \gamma(t) = [v] \in \overline{KM}(X).$$

2. *If $\beta(t)$ is another lift of $\gamma(t)$ to $\mathfrak{D}^+(X)$, then there is a $g \in \Gamma_X$ such that*

$$\beta(t) = exp_{w'}(x_0' + i\,t\,y_0')$$

for $w' = g \cdot w,\ x_0' = g \cdot x_0,\ y_0' = g \cdot y_0$ and $t \gg 0$.

Proof. In [Loo03, 2.2] Looijenga constructs a basis of neighborhoods of $[v] \in KM(X)$ as follows. Let $\Gamma_v = \{g \in \Gamma_X^+ \mid g \cdot v = v\}$. Consider the exponential parametrization

$$exp_v : A(v)_{\mathbb{R}} \times C^+(L(v)) \longrightarrow \mathfrak{D}^+(X).$$

The semi-group $L(v)_{\mathbb{R}} \times C^+(L(v))$ acts on $A(v)_{\mathbb{R}} \times C^+(L(v))$ by translation, and hence also on $\mathfrak{D}^+(X)$. For an open subset K of $\mathfrak{D}^+(X)$ let

$$U(K, v) = \Gamma_v \cdot (L(v)_{\mathbb{R}} \times C^+(L(v))) \cdot K \subset \mathfrak{D}^+(X),$$

which is also an open subset of $\mathfrak{D}^+(X)$.

Then the images of $U(K, v)$ in $KM(X) = \Gamma_X^+ \setminus \mathfrak{D}^+(X)$, where $K \subset \mathfrak{D}^+(X)$ runs through all open and non-empty subsets of $\mathfrak{D}^+(X)$, form a basis of neighborhoods of $[v] \in \overline{KM}(X)$.

Now let $\gamma(t) \in KM(X)$ be a linear degeneration to $[v]$, and $\alpha(t) \in \mathfrak{D}^+(X)$ a lift of $\gamma(t)$ with $\alpha(t) = exp_w(x_0 + ity_0)$. As the vectors v and w are equivalent modulo Γ_X^+ we may assume that $w = v$.

We claim that $\alpha(t) \in U(K, I)$ for all $K \neq \emptyset$ and t sufficiently large. Let $P \in K$ be a point. Write $P = exp_v(r + is)$, then

$$(L(v)_{\mathbb{R}} \times C^+(L(v))) \cdot P = exp_v(r + L(v)_{\mathbb{R}} + i(s + C^+(L(v)))).$$

Clearly $r + L(v)_{\mathbb{R}} = A(v)_{\mathbb{R}}$. Moreover, $y_0 t \in (s + C^+(L(v)))$ for $t \gg 0$ since $(y_0 t - s)^2 > 0$ for $t \gg 0$. This shows that

$$\alpha(t) \in (L(v)_{\mathbb{R}} \times C^+(L(v))) \cdot P \subset U(K, v)$$

for all $t \gg 0$ and therefore proves the claim.

We now proceed to the second claim of the proposition. By assumption, there are $g(t) \in \Gamma_X^+$ such that $\beta(t) = g(t) \cdot \alpha(t)$. We have to show that it is possible to choose $g(t)$ independent of t for $t \gg 0$. Note that, the action of Γ_X on $\mathfrak{D}(X)$ is not fixed point free, and hence the elements $g(t)$ itself may depend on t.

Our strategy is to consider a neighborhood of the cusp $[v]$ where we can replace the action of Γ_X by the action of the stabilizer group Γ_v of the cusp $[v]$. Then we show that the fixed point locus of an element $g \in \Gamma_v$ is either disjoint form $\alpha(t)$ or contains $\alpha(t)$ for all t. This can be used to construct a uniform $g = g(t)$.

Step 1: We may assume $\alpha(t_0) = \beta(t_0)$ *and* $g(t) \in \Gamma_v$ *for all* $t \geq t_0$. Indeed, the neighborhoods $\{U(K, v)\}$ of the Cusp $[v]$ constructed above, are invariant under the action of the Γ_v. Moreover, by [BB66, Thm. 4.9. iv)] there is a sub-basis $\{U(K', v)\}$ of open neighborhoods with the property that

$$g \cdot U(K', v) \cap U(K', v) = \emptyset$$

for all $g \in \Gamma_X^+ \setminus \pm\Gamma_v$.[1]

Choose an element $U = U(K', v)$ in this sub-basis. By 1) proved above, there is a $t_0 > 0$ such that $\alpha(t) \in U$ for all $t \geq t_0$. Let $\beta'(t) = g(t_0)^{-1}\beta(t)$, then $\beta'(t_0) \in U$ as well. Suppose, for a contradiction, that $\beta'(t)$ does not lie in U for all $t \geq t_0$. We set

$$t_1 = \min \{ t \geq t_0 \mid \beta'(t) \notin U \} .$$

We have $\beta'(t_1) \in g' \cdot U$, where $g' = g(t_0)^{-1}g(t_1)$. Since $\beta'(t_1) \notin U$, the sets U and $g' \cdot U$ are disjoint. Now $g' \cdot U$ is open, hence there is an $\varepsilon > 0$ such that $\beta'(t_1 - \varepsilon) \in g' \cdot U$. But $\beta'(t_1 - \varepsilon) \in U$ by construction of t_1, a contradiction.

[1]Note that $g \in \Gamma_X$ and $-g$ induce the same action on $\mathfrak{D}(X)$, but at most one of them is contained in Γ_v.

This shows that

$$\beta'(t) = g'(t) \cdot \alpha(t)$$

for some $g'(t) \in \Gamma_v$ for all $t \geq t_0$. Replacing $\beta(t)$ by $\beta'(t) = g(t_0)^{-1}\beta(t)$ we may assume $g(t) \in \Gamma_v$ and $\alpha(t_0) = \beta(t_0)$, without loss of generality.

Step 2: Consider

$$t_1 = \sup \{ t \geq t_0 \mid \alpha(\tau) = \beta(\tau) \text{ for all } \tau \leq t \} .$$

Suppose, for a contradiction, that $t_1 < \infty$. Let $I = [t_1, t_1 + 1]$ and $K = \alpha(I) \cup \beta(I)$. As K is compact and Γ_v acts properly on $\mathfrak{D}(X)$, the set

$$\Gamma'_v = \{ g \in \Gamma_v \mid g \cdot K \cap K \neq \emptyset \}$$

is finite. The interval I is covered by the closed subsets

$$C(g) = \{ t \in I \mid g \cdot \alpha(t) = \beta(t) \} \subset I,$$

with $g \in \Gamma'_v$.

Step 3: For $g_1, g_2 \in \Gamma'_v$, we have $C(g_1) = C(g_2)$ or $C(g_1) \cap C(g_2) = \emptyset$. Indeed, if $\tau \in C(g_1) \cap C(g_2)$, then $g \cdot \alpha(\tau) = \alpha(\tau)$, where $g = g_1^{-1} g_2$. We need to show that $g \cdot \alpha(t) = \alpha(t)$ for all $t \in I$.

An element $g \in \Gamma_v$ acts on $\alpha(t) = exp_v(x_0 + iy_0\, t)$ as

$$g \cdot exp_v(x_0 + iy_0\, t) = exp_v(g \cdot x_0 + ig \cdot ty_0),$$

for the natural actions of Γ_v on $L(v)$ and $A(v)$. Note that the action on $L(v)$ is linear, so that $g \cdot ty_0 = t\, g \cdot y_0$

This show that $g \cdot \alpha(t) = \alpha(t)$ if and only if

$$g \cdot x_0 = x_0, \quad \text{and} \quad g \cdot y_0 = y_0.$$

This condition is independent of t and therefore shows the claim.

Step 4: There exists a $g_1 \in \Gamma'_v$ with $C(g_1) \neq C(\mathrm{id})$ and $\inf C(g_1) = t_1$. Suppose not, then all $C(g)$ with $C(g) \neq C(id)$ are bounded away from t_1. As there are only finitely many of them, the set $C(\mathrm{id})$ contains an interval of the form $[t_1, t_1 + \varepsilon], \varepsilon > 0$, contradicting our choice of t_1.

Step 5: As $C(g_1)$ is closed, it contains its infimum t_1. Clearly $t_1 \in C(\mathrm{id})$. This shows $C(\mathrm{id}) \cap C(g) \neq \emptyset$ and contradicts Step 3. $\square$

6.2 Linear degenerations of stability conditions

We have the following natural maps from the stability manifold to the Kähler moduli space.

$$\begin{tikzcd}
Stab^{\dagger}(X) \arrow[d, "\pi"] \arrow[dr, "\bar{\pi}"] \arrow[drr, "\tilde{\pi}"] & & \\
\mathcal{P}_0^+(X) \arrow[r, "\theta"] & \mathfrak{D}_0^+(X) \arrow[r] & \Gamma_X^+ \setminus \mathfrak{D}_0^+(X) = KM_0(X)
\end{tikzcd}$$

The goal of this section is to proof the following Theorem.

Theorem 6.3. *Let $[v] \in \overline{KM}(X)$ be a standard cusp and $\sigma(t) \in Stab^\dagger(X)$ be a path in the stability manifold with the property that $\tilde{\pi}(\sigma(t)) \in \overline{KM}(X)$ is a linear degeneration to $[v]$. Let Y be the K3 surface associated to $[v]$ by Ma's theorem 3.10. Then there exist*

1. *a derived equivalence $\Phi : \mathcal{D}^b(Y) \xrightarrow{\sim} \mathcal{D}^b(X)$*
2. *classes $x \in NS(Y)_{\mathbb{R}}$, $y \in \overline{Amp}(Y), y^2 > 0$ and*
3. *a path $g(t) \in \widetilde{Gl}_2^+(\mathbb{R})$*

such that

$$\sigma(t) = \Phi_*(\sigma_Y^*(x, t\, y) \cdot g(t))$$

for all $t \gg 0$.[2]

Moreover, the hearts of $\sigma(t) \cdot g(t)^{-1}$ are independent of t for $t \gg 0$. If $y \in Amp(X)$, then the heart can be explicitly described as the tilt $\mathcal{A}_Y(x, y)$ of $Coh(Y)$.

Proof. By Proposition 6.2, $\tilde{\pi}(\sigma(t))$ is a linear degeneration if and only if

$$\bar{\pi}(\sigma(t)) = exp_w(x_0 + i\, t\, y_0) \in \mathfrak{D}_0^+(X)$$

for some $w \in \Gamma_X \cdot v$, $x_0 \in A(w)_{\mathbb{R}}$, $y_0 \in C(L(w))$.

Note that, w is a standard vector and the Hodge structure on $w^\perp/w$ induced by $\tilde{H}(X, \mathbb{Z})$ is isomorphic to $v^\perp/v$. So, by Theorem 5.14 and Remark 3.11 there is a derived equivalence $\Phi : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ such that $\Phi^H(w) = v_0$, and $\Phi_*\sigma(t) \in Stab^\dagger(Y)$. Hence we may assume, without loss of generality, that $w = v_0$.

Now we claim, that there is a continuous path $g(t) \in \widetilde{Gl}_2^+(\mathbb{R})$ such that

$$\pi(\sigma(t) \cdot g(t)) = Exp_{v_0}(x_0 + i\, t\, y_0) \in \mathcal{P}_0^+(X).$$

Moreover, two such paths $g(t), g'(t)$ differ by an even shift $\Sigma_{2k}, k \in \mathbb{Z}$ (cf. Example 4.4), i.e. $g'(t) = \Sigma_{2k} \circ g(t)$.

Indeed, in Remark 2.6 we constructed a section $q_{v_0} : \mathfrak{D}(X) \to \mathcal{P}(X)$ of the $Gl_2^+(\mathbb{R})$-action on $\mathcal{P}(X)$. Hence, there is a unique $h(t) \in Gl_2^+(\mathbb{R})$ such that $\pi(\sigma(t)) \cdot h(t) = q_{v_0}(exp_{v_0}(x_0 + ity_0)) = Exp_{v_0}(x_0 + ity_0)$. Every choice of a continuous lift $g(t)$ of $h(t)$ to $\widetilde{Gl}_2^+(\mathbb{R})$ has the required property. As $\widetilde{Gl}_2^+(\mathbb{R}) \to Gl_2^+(\mathbb{R})$ is a Galois cover with Galois group $\mathbb{Z}$ acting by even shifts $k \mapsto \Sigma_{2k}$ the latter statement follows.

We choose a $t_0 > 0$ such that $(t_0 y_0)^2 > 2$. Lemma 4.21 shows, that there is an auto-equivalence $\Psi \in \widetilde{W}(X) \subset Aut^\dagger(\mathcal{D}^b(X))$, such that $\Psi_*\sigma(t_0) \in \overline{U}(X)$. We can find another $g \in Gl_2^+(\mathbb{R})$ such that $\Psi_*\sigma(t_0) \cdot g \in \overline{V}(X)$. Since $\pi(\overline{V}(X)) \subset q_{v_0}(\mathfrak{D}(X)) \subset \mathcal{P}(X)$ as well as $\pi(\Psi_*\sigma(t_0)) \in q_{v_0}(\mathfrak{D}(X))$ we see that $g \in ker(\widetilde{Gl}_2^+(\mathbb{R}) \to Gl_2^+(\mathbb{R}))$ and therefore there is a (unique) $k \in \mathbb{Z}$ such that $\Psi_*\sigma(t_0) \cdot \Sigma_{2k} \in \overline{V}(X)$. This allows us to assume, without loss of generality, that $\sigma(t_0) \in \overline{V}(X)$.

[2] The stability condition $\sigma_Y^*(x, y)$ was constructed in Lemma 4.11. If $y \in Amp(X)$, then $\sigma_Y^*(x, y)$ agrees with Bridgeland's stability condition $\sigma_Y(x, y)$ (cf. Definition 4.5.)

By assumption on t_0, we have furthermore $\sigma(t_0) \in \overline{V}_{>2}(X)$. Now Lemma 4.11 shows that

$$\sigma(t_0) = \sigma_X^*(x, t_0 y).$$

We claim that the same holds for all $t \geq t_0$. Indeed, let $\sigma'(t) = \sigma_X^*(x, ty)$, $t \geq t_0$. Then $\sigma(t)$ and $\sigma'(t)$ are two lifts of the path $Exp_{v_0}(x+ity) \in \mathcal{P}_0^+(X)$ to $Stab^\dagger(X)$ with the same value at $t = t_0$. As π is a covering-space we have $\sigma(t) = \sigma'(t)$. This shows the claim and therefore the first part of the proposition.

It remains to show, that the hearts $\sigma_X^*(x, ty)$ are independent of $t \geq t_0$. In the case $y \in Amp(X)$ this follows directly from Remark 4.7. The general case is more involved:

We introduce the symbol $\mathcal{A}(t)$ for the heart, and $\mathcal{P}_t(\phi)$ for the slicing, of the stability stability condition $\sigma(t)$. Let $E \in \mathcal{A}(t_0)$, we have to show, that $E \in \mathcal{A}(t)$ for all $t \geq t_0$. We claim the following statements:

1. If $E \in \mathcal{P}_t((0,1))$ for one $t \geq t_0$, then $E \in \mathcal{P}_\tau([0,1])$ for all $\tau \geq t_0$.

2. If $E \in \mathcal{P}_t(0)$ for one $t \geq t_0$, then $E \in \mathcal{P}_\tau(0)$ for all $\tau \geq t_0$.

3. If $E \in \mathcal{P}_t(1)$ for one $t \geq t_0$, then $E \in \mathcal{P}_\tau(1)$ for all $\tau \geq t_0$.

Once we have shown the claim, we argue as follows. Grouping together Harder–Narasimhan factors in $\sigma(t_0)$ we get an exact triangle[3]

$$\begin{tikzcd} A \arrow[r] & E \arrow[r] & B \arrow[r, dotted] & A \end{tikzcd}$$

with $A \in \mathcal{P}_{t_0}(1)$ and $B \in \mathcal{P}_{t_0}((0,1))$. Let now $t \geq t_0$. By (1) we have $B \in \mathcal{P}_t([0,1])$. Taking Harder–Narasimhan filtration in $\sigma(t)$, yields exact triangles

$$\begin{tikzcd} 0 = B_0 \arrow[rr] & & B_1 \arrow[rr] \arrow[dl] & & B_2 \arrow[rr] \arrow[dl] & & B_3 = B \arrow[dl, "\alpha"] \\ & C_1 \arrow[ul, dotted] & & C_2 \arrow[ul, dotted] & & C_3 \arrow[ul, dotted] & \end{tikzcd}$$

with $C_1 \in \mathcal{P}_t(1), C_2 \in \mathcal{P}_t((0,1)), C_3 \in \mathcal{P}_t(0)$. By (2) we have $C_3 \in \mathcal{P}_{t_0}(0)$, but $B \in \mathcal{P}_{t_0}((0,1))$, therefore $(\alpha : B \to C_3) = 0$ which is only possible if $C_3 = 0$. This means that $B \in \mathcal{P}_t((0,1]) = \mathcal{A}(t)$. By (3) we also have $A \in \mathcal{P}_t(1) \subset \mathcal{A}(t)$, which implies $E \in \mathcal{A}(t)$ since $\mathcal{A}(t)$ is extension closed.

Ad 1: Let $\omega \in Amp(X)$ be an ample class, then also $y + s\omega$ is ample for all $s > 0$. For $s \geq 0, t \geq t_0$, let $\sigma(t,s) = \sigma^*(x, t(y+s\omega)) \in \overline{V}(X)$ (cf. Lemma 4.11). Note that $\sigma(t, 0) = \sigma(t)$.

The property $E \in \mathcal{P}_\sigma((0,1))$ is clearly open in σ. Hence we find $\varepsilon > 0$, with $E \in \mathcal{P}_{t,s}((0,1))$ for all $\varepsilon > s \geq 0$. If $s > 0$, then $\sigma(t,s) \in V(X)$ and the heart $\mathcal{A}(\sigma(t,s))$ is independent of $t \geq t_0$ (cf. Remark 4.7). Therefore, $E \in \mathcal{A}(\sigma(\tau, s))$ for all $\tau \geq t_0, s > 0$. Taking the limit $s \to 0$ we find $E \in \mathcal{P}_\tau([0,1])$, for all $\tau \geq t_0$.

Ad 3: Applying a shift we reduce this statement to (2).

Ad 2: The property $E \in \mathcal{P}_\tau(0)$ is clearly closed in τ. It suffices to show openness. Fix $t \geq t_0$ with $E \in \mathcal{P}_t(0)$ and $1 > \varepsilon > 0$.

Let T be the set of objects which occur as semi-stable factors of E in a stability condition $\sigma(\tau), |t - \tau| \leq \varepsilon$. Then the set T has bounded mass (cf. proof of

[3] We use the symbol $\begin{tikzcd} A \arrow[r, dotted] & B \end{tikzcd}$ for a morphism $A \longrightarrow B[1]$.

[Bri08, Prop. 9.3.]), and therefore the set of Mukai vectors $S = \{ v(A) \mid A \in T \}$ is finite (cf. [Bri08, Lem. 9.3]).

Writing out formula for $Z_t(v)$ as in [Bri08, Sec. 6], we see that $Im(Z_t(v))$ vanishes if and only if $Im(Z_\tau(v))$ vanishes for all $\tau \geq t_0$. It follows that S decomposes as a disjoint union $S = S^0 \cup S'$, where $Im(Z_\tau(v)) = 0$ (or $\neq 0$) for all $\tau \geq t_0$, if $v \in S^0$ (or $v \in S'$ respectively).

As the interval $[t-\varepsilon, t+\varepsilon]$ is compact and S is finite, there exists a $1 > \alpha > 0$ such that $|arg(Z_\tau(v))| > \alpha$ for all $v \in S', |\tau - t| \leq \varepsilon$.

Making ε again smaller, we can assume that

$$E \in \mathcal{P}_\tau((-\alpha, \alpha)) \text{ for all } |t - \tau| \leq \varepsilon. \tag{\#}$$

It follows that all semi-stable factors A of E in stability condition $\sigma(\tau)$ with $|\tau - t| \leq \varepsilon$, have the property that $v(A) \in S^0$. Moreover, as $A \in \mathcal{P}_\tau((-\alpha, \alpha))$ and $arg(Z_\tau(A)) \in \mathbb{Z}$, we find $A \in \mathcal{P}_\tau(0)$ and therefore $E \in \mathcal{P}_\tau(0)$. □

6.3 Limiting hearts

Definition 6.4. Let $\mathcal{C}$ be a category. For a sequence of full subcategories $\mathcal{A}(t) \subset \mathcal{C}, t \gg 0$ define the *limit* to be the full subcategory of $\mathcal{C}$ with objects

$$\lim_{t \to \infty} \mathcal{A}(t) = \{ E \in \mathcal{C} \mid E \in \mathcal{A}(t) \text{ for all } t \gg 0 \}.$$

Theorem 6.5. *Let $[v] \in \overline{KM}(X)$ be a standard cusp, and Y the associated K3 surface. Then, there exist a path $\sigma(t) \in Stab^\dagger(X), t \gg 0$ and an equivalence $\Phi : \mathcal{D}^b(Y) \xrightarrow{\sim} \mathcal{D}^b(X)$ such that*

1. $\lim_{t \to \infty} \tilde{\pi}(\sigma(t)) = [v] \in \overline{KM}(X)$ *and*
2. $\lim_{t \to \infty} \mathcal{A}(\sigma(t)) = \Phi(Coh(Y))$

as subcategories of $\mathcal{D}^b(X)$.

Proof. By Theorem 5.14 there is a derived equivalence $\Phi : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ mapping $[v]$ to $[v_0]$ and $\sigma(t)$ into the distinguished component $Stab^\dagger(Y)$. Hence we may assume, without loss of generality, that $[v] = [v_0]$ and $X = Y$.

Let $\omega \in Amp(X)$ and consider the sequence $\sigma(t) = \sigma_X(-\omega t, \omega t) \in Stab^\dagger(X)$.

As in the proof of Proposition 6.2, we see that $\tilde{\pi}(\sigma(t)) = [exp(-t\omega + it\omega)]$ converges to $[v_0]$. Indeed, if a vector $\beta + it\omega$ lies in a principal open $U(K, v_0)$, then also $t\beta + it\omega \in U(K, v_0)$, since $U(K, v)$ is invariant under the additive action of $L(v_0)_\mathbb{R}$ on $T(N, v_0)$.

The heart $\mathcal{A}_X(t\omega, t\omega)$ consists of objects $E \in \mathcal{D}^b(X)$ with $H^0(E) \in \mathcal{T}(t)$, $H^{-1}(E) \in \mathcal{F}(t)$ and $H^i(E) = 0$ for all $i \notin \{0, -1\}$, where

$$\mathcal{T}(t) = \left\{ A \in Coh(X) \mid A \text{ torsion or } \mu_\omega^{min}(A/A_{tors}) > -t\omega^2 \right\}$$
$$\mathcal{F}(t) = \left\{ A \in Coh(X) \mid A \text{ torsion free and } \mu_\omega^{max}(A) \leq -t\omega^2 \right\}.$$

As $\omega^2 > 0$ every sheaf A lies in $\mathcal{T}(t)$ for t sufficiently large. Similarly, no sheaf A lies in $\mathcal{F}(t)$ for all $t \gg 0$. Thus we find $\lim_{t \to \infty} \mathcal{A}_X(\omega t, \omega t) \cong Coh(X)[1]$. □

6.4 Metric aspects

In this section we will define a natural Riemannian metric on the period domain $\mathfrak{D}(N)$ and show that linear degenerations are geodesics.

Let N be a lattice of signature $(2, \rho)$. The natural action of the real Lie group $G = O(N_\mathbb{R})$ on $\mathbb{P}(N_\mathbb{C})$ induces a transitive action of G on $\mathfrak{D}(N)$. Let $[z] \in \mathfrak{D}(N)$ be a point and let $P \subset N_\mathbb{R}$ be the positive definite subspace spanned by $Re(z)$ and $Im(z)$. The stabilizer of $[z] \in \mathfrak{D}(N)$ is the compact subgroup

$$K_P = \{\, g \in G \mid g \cdot [z] = [z] \,\} \cong SO(P) \times O(P^\perp).$$

Let $\mathfrak{k}_P \subset \mathfrak{g}$ be the Lie algebra of $K_P \subset G$. We can identify the tangent space $T_{[z]}\mathfrak{D}(N)$ with the quotient $\mathfrak{g}/\mathfrak{k}_P$.

As G is semi-simple, the Killing form B on $\mathfrak{g}$ is non-degenerate. Let $\mathfrak{m}_P = \mathfrak{k}_P^\perp$ be the orthogonal complement of $\mathfrak{k}_P$ with respect to B. More explicitly, by [Hel78, III.B.ii] we have $B(X, Y) = \rho \cdot Tr(X \circ Y)$ and

$$\mathfrak{m}_P = \left\{\, X \in \mathfrak{g} \mid X(P) \subset P^\perp,\ X(P^\perp) \subset P \,\right\}.$$

We get a Cartan decomposition $\mathfrak{g} = \mathfrak{k}_P \oplus \mathfrak{m}_P$. The restriction of B to $\mathfrak{m}_P$ is positive definite and induces an invariant Riemannian metric on $\mathfrak{D}(N)$ via the canonical isomorphism $\mathfrak{m}_P \cong T_{[z]}\mathfrak{D}(N)$ (cf. [Hel78, III.7.7.4]).

Now [Hel78, IV.3, Thm. 3.3.iii] shows, that the geodesics of $\mathfrak{D}(N)$ through $[z]$ are given by the images $exp(tX) \cdot [z]$ of the one-parameter sub-groups $\{exp(tX) \mid t \in \mathbb{R}\} \subset G$ with $X \in \mathfrak{m}_P$. We will construct a special $X \in \mathfrak{m}_P$ such that $exp(tX) \cdot [z]$ is a linear degeneration through $[z]$.

Let $v_0 \in N$ be a standard vector and let $x + iy \in T(N, v_0)$ with $[z] = exp_{v_0}(x + iy)$. Recall that $x \in N_\mathbb{R}/v_0\mathbb{R}$ with $x.v_0 = -1$. There is a unique lift x_0 of x to $N_\mathbb{R}$ such that $(x_0)^2 = 0$. Indeed, if $\tilde{x}$ is any lift, then $x_0 = \tilde{x} - \frac{1}{2}(\tilde{x}^2)v_0$ has the required property.

Set $x_1 = -x_0$ and let $U \subset N_\mathbb{R}$ be the hyperbolic plane spanned by (v_0, x_1). We get a one-dimensional Lie sub-algebra

$$\mathfrak{a}(v_0, x) = \mathfrak{so}(U) \subset \mathfrak{g}$$

which depends on the choice of v_0 and x.

Lemma 6.6. *The Lie algebra $\mathfrak{a}(v_0, x)$ is contained in $\mathfrak{m}_P$.*

Proof. Let $R = U^\perp$ and decompose $N_\mathbb{R}$ as a direct sum $N_\mathbb{R} = \langle v_0 \rangle \oplus \langle x_1 \rangle \oplus R$. We write elements of $N_\mathbb{R}$ as column vectors $(a, b, c)^{tr} = av_0 + bv_1 + c$ with $a, b \in \mathbb{R}$ and $c \in R$. We have

$$Exp_{v_0}(x + iy) = (-\frac{1}{2}y^2, -1, iy)^{tr}.$$

The two-plane P is spanned by the vectors $(-\frac{1}{2}y^2, -1, 0)^{tr}$ and $(0, 0, y)^{tr}$. The Lie algebra $\mathfrak{a}(v_0, a)$ consists of all matrices

$$A_\lambda = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

One checks easily, that $A_\lambda(P) \perp P$. The orthogonal complement of P consists of all vectors $\gamma = (-\frac{1}{2}by^2, b, c)^{tr}$ with $c.y = 0$. Therefore,

$$A_\lambda(\gamma) = (\lambda \frac{1}{2} by^2, -\lambda b, 0)^{tr} = \lambda b \alpha \in P.$$

This shows that $\mathfrak{a}(v_0, x) \subset \mathfrak{m}_P$. $\square$

Lemma 6.7. *Let* $[z] = exp_{v_0}(x+iy) \in \mathfrak{D}(N)$ *and* $A_\lambda \in \mathfrak{a}(v_0, x)$*, then the action of* $exp(A_\lambda)$ *is given by*

$$exp(A_\lambda) \cdot [z] = exp_{v_0}(x + ity),$$

where $t = exp(\lambda)$*.*

Proof. As above we write elements of $N_\mathbb{R}$ as column vectors with respect to the decomposition $N_\mathbb{R} = \langle v_0 \rangle \oplus \langle x_1 \rangle \oplus R$. Similarly, endomorphisms are represented by matrices. We have

$$exp(A_\lambda) = \begin{pmatrix} t & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & id_R \end{pmatrix},$$

where $t = exp(\lambda)$. Therefore,

$$\begin{aligned} exp(A_\lambda) \cdot P &= \langle (-t\frac{1}{2}y^2, -t^{-1}, 0)^{tr}, (0, 0, y)^{tr} \rangle \\ &= \langle (-\frac{1}{2}(ty)^2, -1, 0)^{tr}, (0, 0, ty)^{tr} \rangle, \end{aligned}$$

which is the two-plane spanned by the real- and imaginary parts of the vector $Exp_{v_0}(x + ity)$. $\square$

Corollary 6.8. *For all* $x + iy \in T(N, v_0)$*, the path*

$$\alpha(\lambda) = exp_{v_0}(x + i\, exp(\lambda)\, y) \in \mathfrak{D}(N)$$

is a geodesic of constant speed.

If $\Gamma \subset G$ is a discrete subgroup acting properly and discontinuously on $\mathfrak{D}(N)$ then the quotient $\Gamma \setminus \mathfrak{D}(N)$ inherits a Riemannian metric on the smooth part $(\Gamma \setminus \mathfrak{D}(N))_{reg}$. Geodesics in $(\Gamma \setminus \mathfrak{D}(N))_{reg}$ are locally the images of geodesics on $\mathfrak{D}(N)$. More generally we define geodesics in $\Gamma \setminus \mathfrak{D}(N)$ to be the images of geodesics in $\mathfrak{D}(N)$.

This discussion applies in particular to the Kähler moduli space of a K3 surface X. From the definition of linear degeneration and Corollary 6.8 we get immediately the following statement.

Corollary 6.9. *Linear degenerations are geodesics in the Kähler moduli space* $KM(X)$*.*

Note however, that our parametrization $exp(x+ity)$ is not of constant speed. We conjecture the following converse to the above corollary.

Conjecture 6.10. *Let* $[v] \in \overline{KM}(X)$ *be a zero-dimensional cusp. Then every geodesic converging to* $[v]$ *is a linear degeneration.*

We have the following evidence. The conjecture holds true in the case X has Picard rank one. Then, $\mathfrak{D}(N)^+$ is isomorphic to the upper half plane and the geodesics converging to the cusp $i\infty$ are precisely the vertical lines, which are our linear degenerations.

If one uses the reductive Borel–Serre compactification $\overline{KM}(X)^{BS}$ to compactify $KM(X)$, then the analogues conjecture seems to follow from [JM02]. Indeed, Ji and MacPherson describe the boundary of $\overline{KM}(X)^{BS}$ as a set of equivalence classes of, so called, EDM-geodesics (cf. [JM02, Prop. 14.16]). Moreover, all EDM-geodesics are classified in [JM02, Thm. 10.18]. They are of the form $(u, z, exp(tH)) \in N_Q \times X_Q \times A_Q$ where $Q \subset G$ is a rational parabolic subgroup and $N_Q \times X_Q \times A_Q \cong \mathfrak{D}(N)$ is the associated horocycle decomposition. We think, that linear degenerations to $[v]$ are the geodesics associated to the stabilizer group $G_{[v]}$ of $[v] \in \mathbb{P}(N_{\mathbb{C}})$. Moreover, all geodesics γ that converge to the boundary component $e([v]) \subset \overline{KM}(X)^{BS}$ associated to $G_{[v]}$ should have the EDM property. It follows form the classification, that γ is of the form $(u, z, exp(tH))$ for some rational parabolic subgroup $Q \subset G$. Since γ converges to $e([v])$, we have $Q = G_{[v]}$ and therefore γ should be a linear degeneration.

There is a natural map $\overline{KM}(X)^{BS} \to \overline{KM}(X)$ (cf. [BJ06, III.15.4.2]). One should be able to prove the full conjecture by studying the fibers of this map over a cusp $[v] \in \overline{KM}(X)$.

Chapter 7

Moduli spaces of complexes on K3 surfaces

In this chapter we construct K3 surfaces as moduli spaces of stable objects in the derived category of another K3 surface. First we introduce a moduli functor, which is a set-valued version of Lieblich's moduli stack cf. [Lie06]. We will show in section 7.3, that Fourier–Mukai equivalences induce natural isomorphisms between moduli spaces. Finally, in section 7.5 we prove our main theorem.

Before we can give the actual definition, we recall the notion of a perfect complex in the first section. Moreover, we establish a base-change formula and a semi-continuity result which will be important later.

7.1 Perfect complexes

We denote by $\mathcal{D}(X) = \mathcal{D}(Coh(X))$ the unbounded derived category of coherent sheaves on X.

Definition 7.1. Let $X \to T$ be a morphism of schemes. A complex $E \in \mathcal{D}(X)$ is called *relatively T-perfect*, if there is an open cover $\{U_\nu\}$ of X such that $E|_{U_\nu}$ is quasi-isomorphic to a bounded complex of T-flat sheaves of finite presentation.

We call E *strictly T-perfect* if E itself is quasi-isomorphic to a bounded complex of T-flat sheaves of finite presentation.

Lemma 7.2. *[Lie06, Cor. 2.1.7] If T is an affine scheme and $f : X \to T$ is a flat, finitely presented and quasi-projective morphism, then every relatively T-perfect complex is strictly T-perfect.*

The following base-change result is presumably well known to the experts. The main difference to the usual base change theorems like [Har77, Prop. 5.2] is that we do not assume flatness of any maps, but perfectness of the complex.

Proposition 7.3 (Base Change)**.** *Consider a diagram of separated, noetherian schemes*

$$\begin{array}{ccc} X' & \xrightarrow{j} & X \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ Y' & \xrightarrow{i} & Y \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u} \\ S' & \xrightarrow{k} & S \end{array}$$

where p is proper and both squares are Cartesian. Let $E \in \mathcal{D}^b(X)$ be a strictly S-perfect complex. Then the base change morphism

$$\mathbb{L}i^*\mathbb{R}p_*E \longrightarrow \mathbb{R}q_*\mathbb{L}j^*E$$

is an isomorphism.

The same holds true if $E \in \mathcal{D}^b(X)$ is only S-perfect but p is flat, finitely presented and projective.

Proof. As the statement is local in Y' we may assume that Y, Y', S, S' are affine. If E is S-perfect and p is flat, finitely-presented and projective, then Lemma 7.2 shows that E is strictly S-perfect. Hence it suffices to treat the case that E is a bounded complex of S-flat coherent sheaves on X.

Step 0) Choose a finite open affine cover $\mathfrak{U} = \{U_\nu\}$ of X. The Čech-complex $\mathcal{C}(\mathfrak{U}, E) \in \mathcal{D}^b(X)$ of E with respect to $\mathfrak{U}$ is the total complex of the following double complex of quasi-coherent sheaves on X

$$\mathcal{C}^q(\mathfrak{U}, E^p) = \prod_{\nu_0<\cdots<\nu_q} \iota_* E^p|_{U_{\nu_0}\cap\cdots\cap U_{\nu_q}}, \quad d_1^{pq} = d_E^p, d_2^{pq} = (-1)^p\delta^q,$$

where $\iota : U_{\nu_0} \cap \cdots \cap U_{\nu_q} \to X$ denotes the inclusion. It comes with a canonical morphism

$$E \longrightarrow \mathcal{C}(\mathfrak{U}, E), \quad m \in E^p \mapsto (m|_{U_\nu})_\nu \in \mathcal{C}^0(\mathfrak{U}, E^p). \tag{7.1}$$

which is a quasi-isomorphism. This can be checked using the spectral sequence for double complexes and the vanishing of the E_2^{pq} in degrees $q \neq 0$.

Step 1) We claim that the sheaves $\mathcal{C}^n(\mathfrak{U}, E), n \in \mathbb{Z}$ are acyclic for p_*. The sheaf $\mathcal{C}^n(\mathfrak{U}, E)$ is a direct sum of sheaves of the form $\iota_*E^p|_{U'}$ where $U' = U_{\nu_0} \cap \cdots \cap U_{\nu_n}$. Since X is separated U' is affine. The morphism $p' : U' \longrightarrow Y$ between affine schemes is affine and hence all higher direct images $\mathbb{R}^ip'E^p|_{U'} = 0, i > 0$ vanish. Hence

$$\mathbb{R}p_*E \cong p_*\mathcal{C}(\mathfrak{U}, E).$$

The sheaves $\mathcal{C}^n(\mathfrak{U}, E)$ are still S-flat, since $p_*E|_{U'}$ are given by restriction of scalars along the morphism of affine schemes $p' : U' \to Y$. This shows that

$$\mathbb{L}i^*\mathbb{R}p_*E \cong i^*p_*\mathcal{C}(\mathfrak{U}, E). \tag{7.2}$$

Step 2) On the other hand we have

$$\mathbb{L}j^*E \cong j^*\mathcal{C}(\mathfrak{U}, E),$$

since $\mathcal{C}^n(\mathfrak{U}, E)$ is S-flat for all $n \in \mathbb{Z}$.

We claim that $j^*\mathcal{C}^n(\mathfrak{U}, E), n \in \mathbb{Z}$ are acyclic for q_*.

Again we use that $j^*\mathcal{C}^n(\mathfrak{U}, E)$ are direct sums of sheaves of the form $j^*\iota_* E^p|_{U'}$ with $U' = U_{\nu_0} \cap \cdots \cap U_{\nu_q}$. Consider the open affine subset $V' = j^{-1}(U') \subset X'$ and the following diagram

$$\begin{tikzcd}
V' \arrow[r, "j'"] \arrow[d, "\iota'"] \arrow[dd, bend right=60, "q'"'] & U' \arrow[d, "\iota"] \arrow[dd, bend left=60, "p'"] \\
X' \arrow[r, "j"] \arrow[d, "q"] & X \arrow[d, "p"] \\
Y' \arrow[r, "i"] & Y
\end{tikzcd}$$

We have

$$j^*\iota_* E^p|_{U'} = j^*\iota_*\iota^* E^p = \iota'_* j'^* \iota^* E^p = \iota'_* \iota'^* j^* E^p.$$

In the second step we use base change for open inclusions of affine schemes into separated schemes. It follows that the higher direct images vanish:

$$\mathbb{R}^i q_*(j^*\iota_* E^p|_{U'}) = \mathbb{R}^i q_*(\iota'_* \iota'^* j^* E^p) = \mathbb{R}^i q'_*(\iota'^* j^* E^p) = 0$$

for all $i > 0$. We used in the second step that ι_* is exact and in the third step that q' is affine.

This shows that

$$\mathbb{R}q_*\mathbb{L}j^*E \cong q_* j^* \mathcal{C}(\mathfrak{U}, E). \tag{7.3}$$

Step 3) The base change morphism

$$\theta : \mathbb{L}i^*\mathbb{R}p_*E \longrightarrow \mathbb{R}q_*\mathbb{L}j^*E$$

can be constructed using the adjunction of $\mathbb{L}q^*, \mathbb{R}q_*$, and $\mathbb{L}p^*, \mathbb{R}p_*$. It can be computed on appropriate resolutions using the adjunction of functors of sheaves between q^*, q_* and p^*, p_*.

Under the quasi-isomorphisms (7.3) and (7.2) the morphism θ is given by a morphism of complexes whose components are base-change morphisms

$$q_* j^* \iota_* E^p|_{U'} = q'_* j'^* E^p|_{U'} \longrightarrow i^* p'_* E^p|_{U'} = i^* p_* \iota_* E^p|_{U'}$$

for the affine schemes V', U', Y, Y' and thus isomorphisms. □

Proposition 7.4 (Semi-continuity)**.** *Let $X \to T$ be a proper morphism between separated, noetherian schemes and let $E \in \mathcal{D}^b(X)$ be a T-perfect complex.*

For $t \in T$, denote by $i_t : X_t = X \times_T \{t\} \to X$ the inclusion of the fiber and by $E_t = \mathbb{L}i_t^* E$ the derived restriction. Then the function

$$\phi^i : T \to \mathbb{Z} : t \mapsto dim_{k(t)} \mathbb{H}^i(X_t, E_t)$$

is upper semi-continuous.

Proof. The base change theorem 7.3 allows us to adapt the proof in [Har77, Thm. III.12.8] given for a T-flat sheaf E to our more general situation.

We may assume T is affine and E is a bounded complex of T-flat coherent sheaves. By Theorem 7.3 we have

$$\mathbb{H}^i(X, E_t) = \mathcal{H}^i(\mathbb{L}i_t^* \mathbb{R}p_* E)$$

as sheaves on $\{t\} = Spec(k(t))$. As we saw in the proof of this theorem, $\mathbb{R}p_*E$ can be represented by a bounded complex $C \in \mathcal{D}^b(QCoh(T))$ of T-flat, quasi-coherent sheaves. Note that the cohomology sheaves $\mathcal{H}^i(C) \cong \mathbb{R}^i p_* E$ are coherent.

In this situation [Har77, Lem. III.12.3] tells us that there exists a bounded complex $L \in \mathcal{D}^b(T)$, quasi-isomorphic to C with L^i coherent and locally free.

As L^i are in particular T-flat, we can compute $\mathbb{L}i_t^* \mathbb{R}p_* E$ as $i_t^* L$. Now the function $t \mapsto \dim_{k(t)} \mathcal{H}^i(i_t^* L)$ is upper semi-continuous. Indeed, we can write

$$\mathcal{H}^i(i_t^* L) = ker(i_t^* W^i \to i_t^* L^{i+1})$$

where $W^i = coker(L^{i-1} \to L^i)$. Moreover, there is an exact sequence

$$W^i \longrightarrow L^{i+1} \longrightarrow W^{i+i} \longrightarrow 0.$$

Applying the right exact functor i_t^* we obtain

$$0 \longrightarrow \mathcal{H}^i(i_t^* L) \longrightarrow i_t^* W^i \longrightarrow i_t^* L^{i+1} \longrightarrow i_t^* W^{i+1} \longrightarrow 0.$$

Therefore

$$\phi^i(t) = dim_{k(t)}(i_t^* W^i) + dim_{k(t)}(i_t^* W^{i+1}) - dim_{k(t)}(i_t^* L^{i+1}).$$

The function $dim_{k(t)}(i_t^* W^j)$ is upper semi-continuous for all $j \in \mathbb{Z}$ since W^j is coherent and T is noetherian [Har77, Ex. III.12.7.2]. Moreover, $dim_{k(t)}(i_t^* L^{i+1})$ is locally constant. This shows $\phi^i(t)$ is upper semi-continuous. ☐

Let $f : X \to Y$ be a morphism of S-schemes, and $E \in \mathcal{D}^b(X)$ an S-perfect complex. Under certain assumptions on f Grothendieck and Illusie show in [Gro71] that the pushforward $\mathbb{R}f_*E$ is again S-perfect. The definition of S-perfectness used in [Gro71] differs slightly form the one we use. The definitions agree if $X \to S$ is flat, of finite type between locally noetherian schemes (cf. [Lie06, Def. 2.1.1.]).

Theorem 7.5. *[Gro71, SGA 6,III,4.8] Assume that $X \to S$ and $Y \to S$ are flat morphisms of finite type between locally noetherian schemes. Let $g : X \to Y$ be a projective morphism of schemes over S.*

If $E \in \mathcal{D}^b(X)$ is relatively S-perfect, then $\mathbb{R}g_*E$ is relatively S-perfect.

7.2 Moduli functor

Let X be a K3 surface and T be a scheme over $\mathbb{C}$. For a point $t \in T(\mathbb{C})$ we denote by $i_t : X \to X \times T$ the inclusion of the fiber and for a complex $E \in \mathcal{D}^b(X \times T)$ let $E_t = \mathbb{L}i_t^* E$ be the restriction.

Definition 7.6. For $v \in N(X)$ and $\sigma \in Stab(X)$ consider the moduli functor

$$\mathcal{M}_X^\sigma(v) : (Shm/\mathbb{C})^{op} \to (Set), \quad T \mapsto \{E \in \mathcal{D}^b(X \times T) \,|\, (*)\,\}/\sim .$$

Here $(Shm/\mathbb{C})$ is the category of separated schemes of finite type over $\mathbb{C}$.[1] The symbol $(*)$ stands for the following conditions.

1. The complex E is relatively T-perfect.
2. For all $t \in T(\mathbb{C})$ the restriction $E_t \in \mathcal{D}^b(X)$ is σ-stable of Mukai vector $v(E_t) = v$.

The equivalence relation $\sim$ is defined as follows. We have $E \sim E'$ if and only if there is an open cover $\cup T_\nu = T$ of T such that for all ν there is a line bundle $L \in Pic(T_\nu)$ and an even number $k \in 2\mathbb{Z}$ with $E \cong E'[k] \otimes pr_2^* L$ on $X \times T_\nu$.

To a morphism of schemes $i : S \to T$ in $(Shm/\mathbb{C})$ the functor assigns the map

$$\mathbb{L}i_X^* : \mathcal{M}_X^\sigma(v)(T) \longrightarrow \mathcal{M}_X^\sigma(v)(S)$$

sending $E \in \mathcal{D}^b(X \times T)$ to $\mathbb{L}i_X^* E$, where $i_X = \mathrm{id}_X \times i$.

Proof. We need to show that $\mathbb{L}i_X^*$ induces a well defined morphism between the moduli functors. The relative perfectness of $\mathbb{L}i_X^* E \in \mathcal{D}^-(X \times T')$ follows from Lemma 7.2 (2).

For all $s \in S(\mathbb{C})$ it is $(\mathbb{L}i_X^* E)_s$ a σ-stable complex of Mukai vector v, since we have associativity of (derived) pullback:

$$(\mathbb{L}i_X^* E)_s = \mathbb{L}i_s^* \mathbb{L}i_X^* E \cong \mathbb{L}i_t^* E = E_t,$$

where $t = i(s) \in T(\mathbb{C})$. Now E_t is σ-stable complex of Mukai vector v by assumption.

Next we claim that the complex $\mathbb{L}i_X^* E$ has bounded cohomology. Indeed, by S-perfectness $\mathbb{L}i_X^* E$ has locally on $T \times X$ bounded cohomology. Moreover, $S \times X$ is quasi-compact, since we assume S is noetherian. Hence the cohomology of $\mathbb{L}i_X^* E$ is globally bounded.

Furthermore, if $E \sim E'$, then $\mathbb{L}i_X^* E \sim \mathbb{L}i_X^* E'$. Indeed, if $E \cong E'[k] \otimes pr_2^* L$ on an open subset $X \times T_\nu$, then also

$$\mathbb{L}i_X^* E \cong \mathbb{L}i_X^* E'[k] \otimes pr_2^* i^* L$$

on $X \times i^{-1}(T_\nu)$. This follows from $\mathbb{L}i_X^* pr_2^* L = i_X^* pr_2^* L \cong \mathrm{pr}_2^*\, i^* L$ which holds as L is locally free (and hence adapted to $\mathbb{L}i_X^*$) and $\mathrm{pr}_2 \circ i_X = i \circ \mathrm{pr}_2$. □

7.3 Moduli spaces under Fourier–Mukai transformations

Theorem 7.7. *Let $\Phi : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ be a Fourier–Mukai equivalence between two K3 surfaces X and Y, then Φ induces an isomorphism of functors*

$$\mathcal{M}_X^\sigma(v) \xrightarrow{\cong} \mathcal{M}_Y^{\Phi_*\sigma}(\Phi_*^H v)$$

[1] For technical reasons related to Proposition 7.3 we have to restrict ourselves to this subcategory.

Proof. Denote by p_T, q_T and π the projections from $X \times Y \times T$ to $X \times T, Y \times T$ and $X \times Y$, respectively. Let $\mathcal{P} \in \mathcal{D}^b(X \times Y)$ be the Fourier–Mukai kernel of Φ. We claim that the map

$$\mathcal{M}_X^{\sigma}(v)(T) \ni E \mapsto \Phi_T(E) := \mathbb{R}q_{T*}(p_T^* E \otimes^{\mathbb{L}} \pi^* \mathcal{P}) \tag{7.4}$$

induces a natural transformation $\mathcal{M}_X^{\sigma}(v) \to \mathcal{M}_Y^{\Phi_*\sigma}(\Phi_*^H v)$ between the moduli functors. For this we need to check the following properties.

1. The complex $\Phi_T(E)$ is relatively T-perfect.
2. Naturality: For all $i : S \to T \in (Shm/\mathbb{C})$ and $E \in \mathcal{M}_X^{\sigma}(v)(T)$ we have

$$\mathbb{L}i_Y^* \circ \Phi_T(E) = \Phi_S \circ \mathbb{L}i_X^*(E).$$

3. For all $t \in T(\mathbb{C})$ the complex $\Phi_T(E)_t$ is $\Phi_*\sigma$-stable of Mukai vector $\Phi^H(v)$.

Ad 1) First note that p_T^*E is T-perfect. Indeed, if $E|_U$ is a bounded complex of T-flat sheaves, then $(p_T^*E)|_{p^{-1}(U)}$ is again a bounded complex of T-flat sheaves since p_T is flat. Moreover, pullbacks of finitely presented sheaves are again finitely presented.

As $X \times Y$ is a smooth projective scheme, we can represent $\mathcal{P}$ by a bounded complex of coherent, locally free sheaves. Then $\pi^*\mathcal{P}$ has the same property. In particular, $\pi^*\mathcal{P}$ is T-perfect. Since locally-free sheaves are acyclic for $\_ \otimes^{\mathbb{L}} \_$ we have

$$\pi^*\mathcal{P} \otimes^{\mathbb{L}} p_T^*E = \pi^*\mathcal{P} \otimes p_T^*E.$$

If on an open subset $p_T^*E|_U \cong F$ is a bounded complex of T-flat coherent sheaves then $(\pi^*\mathcal{P} \otimes^{\mathbb{L}} p_T^*E)|_U = \pi^*\mathcal{P}|_U \otimes F$ is also a bounded complex of T-flat coherent sheaves. Hence $\pi^*\mathcal{P} \otimes^{\mathbb{L}} p_T^*E$ is T-perfect.

We now apply Theorem 7.5 to see that $\mathbb{R}q_{T*}(p_T^*E \otimes^{\mathbb{L}} \pi^*\mathcal{P})$ is T-perfect.

Ad 2) Let $i : S \to T$ be a morphism in $(Shm/\mathbb{C})$. We get the following diagram induced from i by pullback.

$$\begin{tikzcd}
X \times S \arrow[d, "i_X"] & X \times Y \times S \arrow[l, "p_S"'] \arrow[r, "q_S"] \arrow[d, "k"] & Y \times S \arrow[d, "i_Y"'] \\
X \times T & X \times Y \times T \arrow[l, "q_T"'] \arrow[r, "p_T"] & Y \times T.
\end{tikzcd}$$

Moreover, for any scheme W we denote by $\pi_W : X \times Y \times W \longrightarrow X \times Y$ the projection. We compute

$$\begin{aligned}
\mathbb{L}i_Y^*\Phi_T(E) &= \mathbb{L}i_Y^*\mathbb{R}q_{T*}(p_T^*E \otimes^{\mathbb{L}} \pi_T^*\mathcal{P}) \overset{7.3}{=} \mathbb{R}q_{S*}\mathbb{L}k^*(p_T^*E \otimes^{\mathbb{L}} \pi_T^*\mathcal{P}) \\
&= \mathbb{R}q_{S*}(\mathbb{L}(p_T \circ k)^*E \otimes^{\mathbb{L}} \mathbb{L}(\pi_T \circ k)^*\mathcal{P}) \\
&= \mathbb{R}q_{S*}(\mathbb{L}(i_X \circ p_S)^*E \otimes^{\mathbb{L}} \pi_S^*\mathcal{P}) \\
&= \mathbb{R}q_{S*}(p^*(\mathbb{L}i_X^*E) \otimes^{\mathbb{L}} \pi_S^*\mathcal{P}) \\
&= \Phi_S(\mathbb{L}i_X^*E).
\end{aligned}$$

For the first step we used a non-flat base change for T-perfect complexes which is established in Proposition 7.3.

Ad 3) As we have seen in (2) the inclusion $i : \{t\} \to T$ induces a natural transformation $\Phi_T \to \Phi_t$. This means we have the following compatibility

$$\Phi_T(E)_t \cong \Phi(E_t).$$

Then (3) follows from the definition of $\Phi_*\sigma$ and $\Phi^H(v)$.

Finally we need to show that (7.4) is an isomorphism. For this we use the following Lemma.

Lemma 7.8. *Let $\Phi : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ and $\Psi : \mathcal{D}^b(Y) \to \mathcal{D}^b(Z)$ be derived equivalences between K3 surfaces X, Y, Z, then*

$$\Phi_T \circ \Psi_T = (\Phi \circ \Psi)_T : \mathcal{M}_X^\sigma(v) \longrightarrow \mathcal{M}_Z^\tau(w)$$

where $w = \Psi^H(\Phi^H(v))$ and $\tau = \Psi_*(\Phi_*(\sigma))$.

Given this lemma we conclude as follows. An inverse to the equivalence Φ is given by a Fourier–Mukai transformation Ψ with kernel $\mathcal{P}^\vee[2]$. Moreover, the kernels of the compositions $\Psi\circ\Phi$, $\Phi\circ\Psi$ are quasi-isomorphic to $\mathcal{O}_\Delta \in \mathcal{D}^b(X\times X)$ and $\mathcal{O}_\Delta \in \mathcal{D}^b(Y \times Y)$ respectively (cf. [Huy06, 5.7, ff.]). Clearly $\mathcal{O}_\Delta$ induces the identity on $\mathcal{M}_X^\sigma(v), \mathcal{M}_Y^\sigma(v)$. This shows that Φ_T and Ψ_T are inverse natural transformations. $\square$

Proof of Lemma. This is a straight forward computation using flat base change and the projection formula. We introduce the following notations for the various projection maps.

$$\begin{tikzcd}
& & X \times Z \arrow[dddll, bend right, "\tilde{p}"'] \arrow[dddrr, bend left, "\tilde{s}"] & & \\
& & X \times Y \times Z \arrow[u, "t"] \arrow[dl, "p'"'] \arrow[dr, "s'"] \arrow[ddll, bend right, "p''"'] \arrow[ddrr, bend left, "s''"] & & \\
& X \times Y \arrow[dl, "p"'] \arrow[dr, "q"] & & Y \times Z \arrow[dl, "r"'] \arrow[dr, "s"] & \\
X & & Y & & Z
\end{tikzcd}$$

We also need a version of this diagram where all objects W are replaced by the products $W\times T$ and all morphisms f are replaced by the products $f_T = f\times \mathrm{id}_T$. The projections $W \times T \to W$ are denoted by π_W.

If $\mathcal{P} \in \mathcal{D}^b(X \times Y)$ and $\mathcal{Q} \in \mathcal{D}^b(Y \times Z)$ are Fourier–Mukai kernels of Φ and Ψ respectively, then

$$\mathcal{R} = \mathbb{R}t_*({p'}^*\mathcal{P} \otimes^{\mathbb{L}} {s'}^*\mathcal{Q})$$

is the Fourier–Mukai kernel of $\Psi \circ \Phi$. We denote the pull-back of the kernels by $\mathcal{P}_T = \pi_{X\times Y}\mathcal{P}$,$\mathcal{Q}_T = \pi_{Y\times Z}\mathcal{Q}$ and $\mathcal{R}_T = \pi_{X\times Z}\mathcal{R}$. Note that we also have

$$\begin{aligned}
\mathcal{R}_T &= \pi_{X\times Z}^*\mathbb{R}t_*(s'^*(\mathcal{Q}) \otimes^{\mathbb{L}} {p'_T}^*(\mathcal{P})) \\
&= \mathbb{R}t_{T*}\pi_{X\times Y\times Z}^*(s'^*(\mathcal{Q}) \otimes^{\mathbb{L}} {p'_T}^*(\mathcal{P})) \\
&= \mathbb{R}t_{T*}({s'_T}^*\pi_{Y\times Z}^*(\mathcal{Q}) \otimes^{\mathbb{L}} {p'_T}^*\pi_{X\times Y}^*(\mathcal{P})) \\
&= \mathbb{R}t_{T*}({s'_T}^*\mathcal{Q}_T \otimes^{\mathbb{L}} {p'_T}^*\mathcal{P}_T)
\end{aligned}$$

using flat base change. Now we conclude as follows

$$\begin{aligned}
\Psi_T \circ \Phi_T(E) &= \mathbb{R}s_{T*}(\mathcal{Q}_T \otimes^{\mathbb{L}} r_T^* \mathbb{R}q_{T*}(\mathcal{P}_T \otimes^{\mathbb{L}} p_T^* E)) \\
&= \mathbb{R}s_{T*}(\mathcal{Q}_T \otimes^{\mathbb{L}} \mathbb{R}s'_{T*} {p'_T}^* (\mathcal{P}_T \otimes^{\mathbb{L}} p_T^* E)) \\
&= \mathbb{R}s_{T*} \mathbb{R}s'_{T*}({s'_T}^* \mathcal{Q}_T \otimes^{\mathbb{L}} {p'_T}^* (\mathcal{P}_T \otimes^{\mathbb{L}} p_T^* E)) \\
&= \mathbb{R}s''_{T*}({s'_T}^* \mathcal{Q}_T \otimes^{\mathbb{L}} {p'_T}^* \mathcal{P}_T \otimes^{\mathbb{L}} {p''_T}^* E) \\
&= \mathbb{R}\tilde{s}_{T*} \mathbb{R}t_{T*}({s'_T}^* \mathcal{Q}_T \otimes^{\mathbb{L}} {p'_T}^* \mathcal{P}_T \otimes^{\mathbb{L}} t_T^* \tilde{p}_T^* E) \\
&= \mathbb{R}\tilde{s}_{T*}(\mathbb{R}t_{T*}({s'_T}^* \mathcal{Q}_T \otimes^{\mathbb{L}} {p'_T}^* \mathcal{P}_T) \otimes^{\mathbb{L}} \tilde{p}_T^* E) \\
&= \mathbb{R}\tilde{s}_{T*}(\mathcal{R}_T \otimes^{\mathbb{L}} \tilde{p}_T^* E) \\
&= (\Psi \circ \Phi)_T(E). \qquad \square
\end{aligned}$$

7.4 More on stability conditions

Before we can finally state our main result on moduli spaces of stable objects we need another digression on stability conditions. First, we prove a classification result for semi-stable objects, then we introduce v_0-general stability conditions and derive some basic properties.

Proposition 7.9. *Let $\sigma \in U(X)$ be a stability condition. Then an object E is σ-semi-stable with Mukai vector $v_0 = (0,0,1)$ if and only if there is an $x \in X$ and $k \in 2\mathbb{Z}$ such that $E \cong \mathcal{O}_x[k]$.*

$$\left\{ E \in \mathcal{D}^b(X) \mid v(E) = v_0, E\ \sigma\text{-semi-stable} \right\} = \{ \mathcal{O}_x[2k] \mid x \in X, k \in \mathbb{Z} \}.$$

Proof. Let $\sigma \in U(X)$ be a stability condition. The objects $\mathcal{O}_x, x \in X$ are σ-stable by Theorem 4.9 and hence in particular semi-stable.

Let E' be a σ-semi-stable object with Mukai vector v_0. Applying an element in $\widetilde{Gl}_2^+(\mathbb{R})$ we can assume that σ is of the form $\sigma(\omega,\beta)$. There is a unique $k \in \mathbb{Z}$ such that $E'[k] = E$ lies in the heart $\mathcal{A}(\omega,\beta)$. As $Z_\sigma(E) = (-1)^k Z_\sigma(v_0) = -(-1)^k$ has to lie in $\mathbb{H} \cup \mathbb{R}_{<0}$ the number k has to be even and the phase of E is one. Take a Jordan–Hölder filtration

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = E$$

of E in $\mathcal{A}(\omega,\beta)$. The stable quotients $A_i = E_i/E_{i-1}$ have the same phase as E. Hence we can use the classification result of Huybrechts, [Huy08, Prop. 2.2], which shows that $A_i = F[1]$ for a vector bundle F or $A_i = \mathcal{O}_x$ for some $x \in X$ Note that the Mukai vectors in these two cases are given by

$$v(F[1]) = -(r,l,s) \text{ with } r > 0, \quad v(\mathcal{O}_x) = (0,0,1).$$

By assumption we have $\sum_i v(A_i) = v(E) = (0,0,1)$. Hence the sum over the ranks of all occurring vector bundles has to be zero. This is only possible if there are none of them. Hence E is an extension of skyscraper sheaves. Comparing Mukai vectors again, one sees that E has to be of the form $\mathcal{O}_x$ for some $x \in X$. $\square$

Definition 7.10. Fix a Mukai vector $v \in N(X)$. A stability condition $\sigma \in Stab^\dagger(X)$ is called *v-general* if every σ-semi-stable object E of Mukai vector $v(E) = v$ is σ-stable.

Lemma 7.11. *Every stability condition $\sigma \in U(X)$ is $v_0 = (0,0,1)$-general. No stability condition $\sigma \in \partial U(X) \subset Stab^\dagger(X)$ is v_0-general.*

Proof of Lemma. By Proposition 7.9 all σ-semi-stable objects of Mukai vector v_0 are shifts of skyscraper sheaves. All skyscraper sheaves $\mathcal{O}_x$ are σ-stable by Proposition 4.9.

For the second claim note that $\mathcal{O}_x$ remains semi-stable for $\sigma \in \overline{U}(X)$. If $\sigma \in \overline{U}(X)$ and all $\mathcal{O}_x$ are σ-stable, then $\sigma \in U(X)$ by Proposition 4.9. Hence for $\sigma \in \partial U(X)$ there are strictly semi-stable skyscraper sheaves. This means σ is not v_0-general. ☐

Lemma 7.12. *For all primitive Mukai vectors $v \in N(X)$ the set of v-general stability conditions is dense and open in $Stab^\dagger(X)$.*

Proof. Choose an open subset B° with compact closure B. By Lemma 4.14, the set

$$S = \{E \in \mathcal{D}^b(X) \,|\, E \ \sigma\text{-semi-stable for some } \sigma \in B, v(E) = v\}$$

has bounded mass. So by 4.13 we find finitely many codimension one submanifolds $W_\gamma, \gamma \in \Gamma$ of B such that the complement

$$B \setminus \bigcup_{\gamma \in \Gamma} W_\gamma$$

consists of v-general stability conditions. Clearly this subset is dense in B.

The openness follows directly from [Bri08, Prop. 9.4] applied to S. ☐

7.5 Reconstruction theorem

Theorem 7.13. *Let $v \in N(X)$ be a standard vector and $\sigma \in Stab^\dagger(X)$ a v-general stability condition.*

1. *There exists a K3 surface Y and an isomorphism of functors*

$$\mathcal{M}_X^\sigma(v) \cong \underline{Y}$$

   where $\underline{Y}$ is the functor $(Shm/\mathbb{C})^{op} \to (set) : T \mapsto Mor(T, Y)$.

2. *The Hodge structure $H^2(Y, \mathbb{Z})$ is isomorphic to the subquotient of $\tilde{H}(X, \mathbb{Z})$ given by $v^\perp/v$.*

3. *The universal family $E \in \mathcal{M}_X^\sigma(v)(Y) \subset \mathcal{D}^b(X \times Y)$ induces a derived equivalence $\mathcal{D}^b(X) \to \mathcal{D}^b(Y)$.*

Proof. The proof consists of three steps. First, we treat the case $v = v_0, \sigma \in U(X)$ and show that $\mathcal{M}_\sigma(v_0) \cong \underline{X}$ using Proposition 7.9. Next we generalize to $v = v_0$ and $\sigma \in Stab^\dagger(X)$ using Lemma 4.21. Finally the general case can be reduced to $v = v_0$ using Theorem 5.14.

Step 1: Assume that $v_0 = (0,0,1) \in N(X)$, and $\sigma \in U(X)$, then

$$\mathcal{M}^\sigma(v_0) \cong \underline{X}.$$

Indeed, the morphism $\underline{X} \to \mathcal{M}^\sigma_X(v_0)$ is given by

$$f : T \to X \quad \mapsto \quad \mathcal{O}_{\Gamma_f} \in \mathcal{M}^\sigma_X(v_0)(T)$$

where $\Gamma_f \subset X \times T$ is the graph of f. We have to show this map is an isomorphism.

Injectivity: If we have two morphisms $f, g : T \to X$ with $\mathcal{O}_{\Gamma_f} \sim \mathcal{O}_{\Gamma_g}$ then we claim that $f = g$. Indeed, by assumption there is a quasi-isomorphism

$$\mathcal{O}_{\Gamma_f} \cong \mathcal{O}_{\Gamma_g}[k] \otimes \mathrm{pr}_2^* L \quad \text{in} \quad \mathcal{D}^b(X \times T)$$

for some $k \in 2\mathbb{Z}, L \in Pic(T)$. As $\mathcal{O}_{\Gamma_f}$ and $\mathcal{O}_{\Gamma_g} \otimes \mathrm{pr}_2^* L$ are sheaves, we have $k = 0$ and the quasi-isomorphism is an isomorphism of coherent sheaves. Moreover,

$$L = \mathrm{pr}_{2*}(\mathcal{O}_{\Gamma_g} \otimes \mathrm{pr}_2^* L) \cong \mathrm{pr}_{2*}(\mathcal{O}_{\Gamma_f}) = \mathcal{O}_T.$$

Hence it is $\mathcal{O}_{\Gamma_f} \cong \mathcal{O}_{\Gamma_g}$ and it follows that $f = g$.

Surjectivity: If $[E] \in \mathcal{M}^\sigma_X(v_0)(T)$, then $\mathbb{L}i_t^* E$ is σ-stable of Mukai vector v_0. It follows from Proposition 7.9 that $\mathbb{L}i_t^* E \cong \mathcal{O}_x[k]$ for a point $x \in X$ and $k \in 2\mathbb{Z}$ depending on t.

We claim that the shift $k \in 2\mathbb{Z}$ is independent of t in each connected component $T_0 \subset T$. Indeed, the function

$$\phi^i : T_0 \to \mathbb{Z} : t \mapsto h^i(X, \mathbb{L}i_t^* E)$$

is upper semi-continues by Proposition 7.4.

As $E_t = \mathbb{L}i_t^* E \cong \mathcal{O}_x[k]$, the function ϕ^i takes values in $\{0, 1\}$ and $\phi^i(t) = \{1\}$ if and only if $k = -i$. Therefore T_0 is the disjoint union of the closed subsets $\{\phi^i(t) \geq 1\}, i \in \mathbb{Z}$ of T_0. As T_0 is connected, there is a unique i such that $\phi^i(t) = 1$ for all $t \in T_0$ and $\phi^j(t) = 0$ for all $j \neq i, t \in T_0$. This shows the claim.

We can now apply Proposition 7.14 proved below to conclude that $E \sim \mathcal{O}_{\Gamma_f}$ for some $f : T \to X$.

Step 2. Assume that $\sigma \in Stab^\dagger(X)$ is a v_0-general stability condition. Then $\mathcal{M}^\sigma(v_0)$ is isomorphic to X.

Indeed, by Lemma 4.21 we find a $\Phi \in \widetilde{W}(X)$ such that $\sigma' = \Phi_*(\sigma) \in \overline{U}(X)$. Also note that we have

$$T^H_{\mathcal{O}_C(k)}(v_0) = v_0, \quad {T_A^2}^H(v_0) = v_0.$$

It follows that v_0-general stability conditions are mapped to v_0-general stability conditions. By Lemma 7.11 we conclude that $\sigma' \in U(X)$ and not in $\partial U(X)$.

As we have seen in Theorem 7.7 the Fourier–Mukai equivalence Φ^{-1} induces isomorphisms of functors

$$\mathcal{M}^\sigma(v_0) \cong \mathcal{M}^{\Phi_*\sigma}(\Phi^H_* v_0)) = \mathcal{M}^{\sigma'}(v_0) \cong \underline{X}.$$

The last isomorphism is provided by step 1.

Step 3. General case. Let v be a standard vector and $\sigma \in Stab^\dagger(X)$ be a v-general stability condition. By Theorem 5.14 there is a K3 surface Y with Hodge structure $H^2(Y, \mathbb{Z}) \cong v^\perp/v$ and a derived equivalence $\Phi : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ respecting the distinguished component and mapping $v \in N(X)$ to $v_0 \in N(Y)$.

Since $\sigma \in Stab^\dagger(X)$ is v-general also $\Phi_*(\sigma) \in Stab^\dagger(Y)$ is v_0-general. By Theorem 7.7 the Fourier–Mukai transformation $\Phi : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ induces an isomorphism of moduli functors $\mathcal{M}_X^\sigma(v) \cong \mathcal{M}_Y^{\Phi_*\sigma}(v_0)$. Now we apply step 2 to conclude that $\mathcal{M}_Y^{\Phi_*\sigma}(v_0) \cong \underline{Y}$.

It remains to show that the universal family $E \in \mathcal{D}^b(X \times Y)$ induces a derived equivalence. Indeed, let T be a smooth projective scheme over $\mathbb{C}$. We observe that if a complex $K \in \mathcal{M}_X^\sigma(v)(T)$ induces a Fourier–Mukai transformation

$$FM(K) : \mathcal{D}^b(T) \longrightarrow \mathcal{D}^b(X), \quad E \mapsto \mathbb{R}pr_{1*}(K \otimes^{\mathbb{L}} pr_2^* E)$$

then the image $\Phi_T(K) \in \mathcal{M}_Y^{\Phi_*\sigma}(v)(T)$ induces the transformation

$$FM(\Phi_T(K)) = \Phi \circ FM(K) : \mathcal{D}^b(T) \longrightarrow \mathcal{D}^b(Y).$$

This follows directly from the definition of Φ_T and the composition law for Fourier–Mukai kernels.

Recall that the construction of the isomorphism $\underline{Y} \cong \mathcal{M}_X^\sigma(v)$ starts from some $\sigma' \in U(Y)$ and $\mathcal{M}_Y^{\sigma'}(v_0) \cong \underline{Y}$ established in step 1 and then applies a derived equivalence $\Psi : \mathcal{D}^b(Y) \to \mathcal{D}^b(X)$ which maps σ' to σ.

The isomorphism $\mathcal{M}_Y^{\sigma'}(v_0) \cong \underline{Y}$ is induced by the structure sheaf of the diagonal $\mathcal{O}_\Delta \in \mathcal{M}_Y^{\Phi_*\sigma}(v_0)(Y)$. The induced Fourier–Mukai transformation is the identity and hence an equivalence. Now our observation shows that the universal family $E = \Psi_Y(\mathcal{O}_\Delta) \in \mathcal{M}_X^\sigma(v)(Y)$ induces the equivalence $\Psi : \mathcal{D}^b(Y) \to \mathcal{D}^b(X)$. □

In the proof we used the following proposition:

Proposition 7.14. *Let $E \in \mathcal{D}^b(X \times T)$ be a complex such that for all $t \in T(\mathbb{C})$ there is an $x \in X(\mathbb{C})$ with $\mathbb{L}i_t^*(E) \cong \mathcal{O}_x$.*

Then there exists a morphism $f : T \to X$ and a line bundle $L \in T$ such that

$$E \cong \mathcal{O}_{\Gamma_f} \otimes \mathrm{pr}_2^*(L).$$

Proof. By assumption $\mathbb{L}i_t^*(E)$ is a complex concentrated in degree zero, i.e. a sheaf. If follows from [Huy06, Lem. 3.31] that E is a also a sheaf which is moreover flat over T. Let $Z = supp(E)$, and $p : Z \to T$ the projection. We have $p^{-1}(t) = supp(\mathbb{L}i_t^*(E)) = \{x\}$ by Lemma 3.29, loc. cit. We conclude that p is a finite morphism, as it is projective and quasi-finite. (By Lemma 7.15 below, it suffices to check quasi-finiteness on $\mathbb{C}$-valued points).

As E is flat over T and p is finite, the sheaf $p_*(E) \in Coh(T)$ is locally free. Moreover, the fibers are given by

$$p_*(E) \otimes \mathcal{O}_t = p_*(E \otimes p^*\mathcal{O}_t) = \mathcal{O}_t$$

and hence $L = p_*(E)$ is a line bundle.

The $\mathcal{O}_Z$-module structure of E is determined by a morphism

$$m : p_*\mathcal{O}_Z \longrightarrow End(L) = \mathcal{O}_T$$

It is easy to see that the morphisms of sheaves

$$p^* : \mathcal{O}_T \longrightarrow p_*\mathcal{O}_Z \quad m : p_*\mathcal{O}_Z \longrightarrow \mathcal{O}_T$$

are inverse to each other. Taking $\underline{Spec}$ we see that $p = \underline{Spec}(p^*) : Z \to T$ is an isomorphism with inverse $\underline{Spec}(m)$. It remains to check that $p^*L = E$. We have an adjunction-unit $p^*L = p^*p_*E \to E$ which pushes down to

$$m \otimes id : p_*E \otimes p_*\mathcal{O}_Z = p_*p^*(p_*E) \longrightarrow p_*E$$

by definition. As $m \otimes id$ is an isomorphism and p is affine we are done. □

Lemma 7.15. *Let $f : X \to Y$ be a morphism of schemes of finite type over $\mathbb{C}$. If $f^{-1}(y)$ is finite for all $y \in Y(\mathbb{C})$, then $f^{-1}(\eta)$ is finite for all (non-closed) points $\eta \in Y$.*

Proof. We can assume $X = Spec(A), Y = Spec(B)$ are affine. Let $\eta : Spec(K) \to X$ be an arbitrary point of Y and $y : Spec(\mathbb{C}) \to X$ a closed point in the closure of η with ideal sheaf $\mathfrak{m} \subset A$. Such a y exists because X is of finite type over $\mathbb{C}$ and hence Jacobson [Gro67, Cor. 10.4.8].

It is $B \otimes \mathbb{C}$ a finite $\mathbb{C}$-vector space by assumption. Let b'_i be a basis and choose lifts $b_i \in B$ mapping to $b'_i \in B \otimes \mathbb{C}$. By Nakajama's lemma b_i generate $B_\mathfrak{m}$ as an A-module. Hence the images of b_i generate $B_\mathfrak{m} \otimes K = B \otimes K$ as a K-vector space. This means $f^{-1}(\eta)$ is finite. □

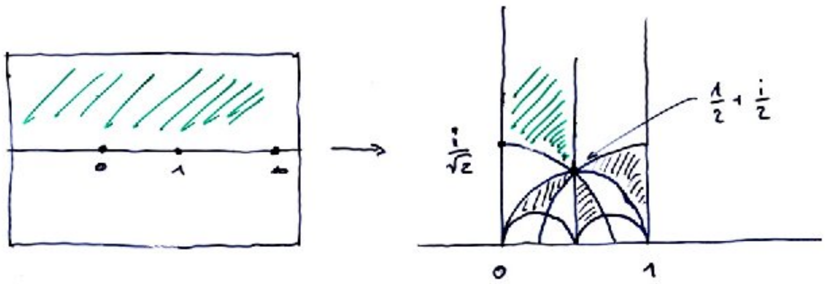
Remark 7.16. In general one expects that the moduli space $\mathcal{M}^\sigma(v)$ undergoes (birational) transformations, called wall-crossings when σ moves in $Stab^\dagger(X)$. This behavior can be observed in our situation, too, but the transformations turn out to be isomorphisms.

If $\sigma \in U(X)$, then $\mathcal{M}^\sigma(v_0)$ parametrizes the skyscraper sheaves $\mathcal{O}_x, x \in X$. When σ passes over a wall of type (C_k), then the sheaves $\mathcal{O}_x, x \in C$ are replaced by the complexes $T_{\mathcal{O}_C(k)}\mathcal{O}_x$, whereas the sheaves $\mathcal{O}_x, x \notin C$ remain stable.

If σ moves over an $(A)^\pm$-type wall, then all sheaves $\mathcal{O}_x$ are replaced by the spherical twists $T_A^{\pm 2}\mathcal{O}_x$.

Part II

Period- and mirror-maps for the quartic K3



Chapter 8

Mirror symmetry for K3 surfaces

In this chapter we summarize Aspinwall and Morrison's description [AM97] of mirror symmetry for K3 surfaces in terms of Hodge structures. Their constructions have been generalized to higher dimensional hyperkähler manifolds by Huybrechts in [Huy04] and [Huy05].

8.1 The classical Hodge structure of a complex K3 surface

Recall the following basic facts about K3 surfaces:

The second cohomology $H^2(X,\mathbb{Z})$ endowed with the cup-product pairing $(a.b) = \int a \cup b$ is an even, unimodular lattice of rank 22 isomorphic to the K3 lattice $\Lambda := 2E_8(-1) \oplus 3U$. The group $H^{2,0}(X) = H^0(X, \Omega_X^2)$ is spanned by a the class of a holomorphic two form Ω which is nowhere vanishing. This class satisfies the properties

$$(\Omega.\Omega) = 0 \qquad (\Omega.\bar{\Omega}) > 0.$$

Remark 8.1. The Hodge structure on $H^2(X,\mathbb{Z})$ is completely determined by the subspace $H^{2,0}(X) \subset H^2(X,\mathbb{C})$. Indeed, we have

$$H^{0,2}(X) = \overline{H^{2,0}(X)} \quad \text{and} \quad H^{1,1}(X) = (H^{2,0}(X) \oplus H^{0,2}(X))^{\perp}.$$

The global Torelli theorem states that a K3 surface is determined up to isomorphy, by it's Hodge structure.

Theorem 8.2 (Piatetski-Shapiro–Shafarevich, Burns–Rapoport)**.** *Two K3 surfaces X, X' are isomorphic if and only if there is a Hodge isometry $H^2(X,\mathbb{Z}) \cong H^2(X',\mathbb{Z})$.*

8.2 CFT-Hodge structures of complex K3 surfaces

There is another weight-two Hodge structure associated to a K3 surface, which plays an important role for mirror symmetry.

Recall that the *Mukai pairing* on the total cohomology $H^*(X,\mathbb{Z})$ by

$$((a_0,a_2,a_4).(b_0,b_2,b_4)) := \int a_2 \cup b_2 - a_0 \cup b_4 - a_4 \cup b_2. \tag{8.1}$$

We denote this lattice by $\tilde{H}(X,\mathbb{Z})$. It is an even, unimodular lattice of rank 24 and signature $(4,20)$ isomorphic to the enlarged K3 lattice $\tilde{\Lambda} := \Lambda \oplus U$.

We define a weight-two Hodge structure on $\tilde{H}(X,\mathbb{Z})$ by setting $H^{2,0}_B(X) = H^{2,0}(X)$ and using the construction in Remark 8.1. Note that

$$H^{1,1}_B(X) = H^0(X,\mathbb{C}) \oplus H^{1,1}(X) \oplus H^4(X,\mathbb{C}).$$

We call $H_B(X,\mathbb{Z}) = (\tilde{H}(X,\mathbb{Z}), (\,.\,), H^{p,q}_B(X))$ the *B-model Hodge structure* of X. The name is motivated by the statement in [AM97], that the "B-model conformal field theory" associated to X is uniquely determined by $H_B(X,\mathbb{Z})$.

One very important occurrence of this Hodge structure is the following theorem.

Theorem 8.3 (Derived global Torelli; Orlov [Orl97])**.** *Two projective K3 surfaces X, X' have equivalent derived categories $\mathcal{D}^b(X) \cong \mathcal{D}^b(X')$ if and only if there exists a Hodge isometry $H_B(X,\mathbb{Z}) \cong H_B(X',\mathbb{Z})$.*

8.3 CFT-Hodge structures of symplectic K3 surfaces

Every Kähler form ω on a K3 surface X defines a symplectic structure on the underlying differentiable manifold. In this chapter we will associate a Hodge structure to this symplectic manifold. Moreover, we shall allow twists by a so called B-field $\beta \in H^2(X,\mathbb{R})$ to get a complexified version.

Given ω and β we define the following class of mixed, even degree

$$z = \exp(i\omega + \beta) = (1, i\omega + \beta, (i\omega+\beta)^2/2) \in \tilde{H}(X,\mathbb{C}). \tag{8.2}$$

This class enjoys formally the same properties as $\Omega \in H^{2,0}(X)$ above:

$$(z.z) = 0, \quad (z.\overline{z}) > 0$$

with respect to the Mukai-pairing. Hence, we can define a Hodge structure $H_A(X,\mathbb{Z})$ on $\tilde{H}(X,\mathbb{Z})$ by demanding $H^{2,0}_A(X) := \mathbb{C}\, z$ via Remark 8.1.

We call $H_A(X,\mathbb{Z})$ the *A-model Hodge structure* of (X,ω,β). Again, the name is motivated by the statement in [AM97], that the "A-model conformal field theory" associated to (X,ω,β) is uniquely determined by $H_A(X,\mathbb{Z})$.

8.4 Mirror symmetries

Two Calabi–Yau manifolds X, Y form a mirror pair if the B-model conformal field theory associated to X is isomorphic to the A-model conformal field theory associated to Y. This motivates the following definition.

Definition 8.4. A complex K3 surface X with holomorphic two-form Ω and a symplectic K3 surface Y with complexified Kähler form $z = \exp(i\omega + \beta)$ form a *mirror pair* if there exists a Hodge isometry

$$H_B(X, \mathbb{Z}) \cong H_A(Y, \mathbb{Z}).$$

Thus a naive translation of Kontsevich's homological mirror conjecture reads as follows.

Conjecture 8.5. *Let X be a K3 surface with holomorphic two-form Ω and Y a K3 surface with Kähler form ω. Then there is an exact equivalence of triangulated categories*

$$\mathcal{D}^b(Coh(X)) \cong \mathcal{D}^\pi(Fuk(Y))$$

if and only if there is a Hodge isometry $H_B(X, \mathbb{Z}) \cong H_A(Y, \mathbb{Z})$.

Note that this is perfectly consistent with Orlov's derived global Torelli theorem.

8.5 Relation to mirror symmetry for lattice polarized K3 surfaces

In this section we compare the Hodge theoretic notion of mirror symmetry to Dolgachev's version for families of lattice polarized K3 surfaces [Dol96]. See also [Huy04, Sec. 7.1] and [Roh04, Sec. 2].

Let $M \subset \Lambda$ be a primitive sublattice. A *M-polarized K3 surface* is a K3 surface X together with a primitive embedding $i : M \to Pic(X)$. We call (X, i) pseudo-ample polarized if $i(M)$ contains a numerically effective class of positive self intersection.

Assume that M has the property, that for any two primitive embeddings $i_1, i_2 : M \to \Lambda$ there is an isometry $g \in O(\Lambda)$ such that $i_2 = g \circ i_1$. Then, there is a coarse moduli space K_M of pseudo-ample M-polarized K3 surfaces.

Fix a splitting $M^\perp = U \oplus \hat{M}$. The above condition ensures, that the isomorphism class of $\hat{M}$ is independent of this choice.

Definition 8.6. The *mirror moduli space of K_M is $K_{\hat{M}}$.*

Symplectic structures on a K3 surface Y in $K_{\hat{M}}$ correspond to points of the mirror moduli space K_M in the following way:

Let $(Y, j) \in K_{\hat{M}}$ be an $\hat{M}$-polarized K3 surface with a marking, i.e. an isometry $n : H^2(Y, \mathbb{Z}) \to \Lambda$, such that $j = n^{-1}|_{\hat{M}}$. Let $\omega + i\beta \in H^2(Y, \mathbb{C})$ be a complexified symplectic structure on Y, which is compatible with the $\hat{M}$-polarization, i.e. $\omega + i\beta \in j(\hat{M})_\mathbb{C}$. Denote by $z = exp(i\omega + \beta)$ the associated period vector.

The chosen splitting $M^\perp = \hat{M} \oplus U$ determines an isometry of $\xi \in O(\tilde{H}(Y,\mathbb{Z}))$ which interchanges the hyperbolic plane $n(U)$ with $H^0(Y,\mathbb{Z}) \oplus H^4(Y,\mathbb{Z})$ and leaves the orthogonal complement fixed.

By construction the vector $\Omega := \xi(z)$ lies in $n(U)_\mathbb{C} \oplus \hat{M}_\mathbb{C} \subset H^2(Y,\mathbb{C})$. Note that $(\Omega.\Omega) = 0$ and $(\Omega.\bar{\Omega}) > 0$. Hence, by the surjectivity of the period map [BBD85, Exp. X], there exists a complex K3 surface X and a isometry $g : H^2(Y,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ that maps Ω into $H^{2,0}(X)$. Extend g to an isometry of Mukai lattices $\tilde{g}$, then

$$\tilde{g} \circ \xi : H_A(Y,\mathbb{Z}) \longrightarrow H_B(X,\mathbb{Z})$$

is an Hodge isometry. Moreover, the marking of Y induces an M-polarization of X via

$$i : M \subset \Lambda \xrightarrow{n} H^2(Y,\mathbb{Z}) \xrightarrow{g} H^2(X,\mathbb{Z}).$$

This means (X,i) lies in the mirror moduli space K_M.

Conversely, if $\Omega \in H^{2.0}(X)$ is the period vector of a marked M-polarized K3 surface, then $z = \xi(\Omega)$ lies in $H^0(X,\mathbb{C}) \oplus \hat{M}_\mathbb{C} \oplus H^4(X,\mathbb{C})$. Hence z is of the form

$$z = a\, exp(i\omega + \beta)$$

for some $\omega, \beta \in M_\mathbb{R}$, $a \in \mathbb{C}^*$. Indeed, write $z = (a,c,b)$ with respect to the above decomposition, then $-2ab + c^2 = 0$ since $z^2 = 0$. Therefore $a \neq 0$ and we can set $i\omega + \beta := c/a \in M_\mathbb{C}$.

Note that $\omega^2 > 0$ since $z.\bar{z} > 0$. Now assume, that ω is represented by a symplectic form, then $i\omega + \beta$ defines a complexified symplectic structure on $Y = X$ such that

$$H_B(X,\mathbb{Z}) \cong H_A(Y,\mathbb{Z}).$$

8.6 Period domains

In order to compare Hodge structures on different manifolds, it is convenient to introduce the period domains classifying Hodge structures.

Let $(L, (\,.\,))$ be a lattice. The *period domain* associated to L is the complex manifold

$$\mathscr{D}(L) := \left\{ [\Omega] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (\Omega.\Omega) = 0,\ (\Omega.\bar{\Omega}) > 0 \right\}.$$

The orthogonal group $O(L, (\,.\,))$ acts on $\mathscr{D}(L)$ from the left.

The period domain carries a tautological variation of Hodge structures on the constant local system L. Indeed, the holomorphic vector bundle $L \otimes \mathcal{O}_{\mathscr{D}(L)}$ has a tautological sub-vector bundle $\mathcal{F}^2$ with fiber $\mathbb{C}\,\Omega \subset L \otimes \mathbb{C}$ over a point $[\Omega] \in \mathscr{D}(L)$. The Hodge filtration is determined by $\mathcal{F}^2$ via

$$\mathcal{F}^2 \subset \mathcal{F}^1 := (\mathcal{F}^2)^\perp \subset L \otimes \mathcal{O}_{\mathscr{D}(L)}. \tag{8.3}$$

8.7 Periods of marked complex K3 surfaces

Let $\pi : X \to B$ be a smooth family of K3 surfaces. We have a local system

$$\mathcal{H}_\mathbb{Z} = \mathbb{R}^2\pi_*\underline{\mathbb{Z}}_X$$

on B with stalks isomorphic to the cohomology $H^2(X_t, \mathbb{Z})$ of the fiber $X_t = \pi^{-1}(\{t\})$. It carries a quadratic form $(\,.\,) : \mathcal{H}_{\mathbb{Z}} \otimes \mathcal{H}_{\mathbb{Z}} \to \mathcal{H}_{\mathbb{Z}}$ and a holomorphic filtration

$$\mathcal{F}^2 = \pi_* \Omega^2_{X/B} \subset \mathcal{F}^1 := (\mathcal{F}^2)^{\perp} \subset \mathcal{H} := \mathcal{H}_{\mathbb{Z}} \otimes \mathcal{O}_B$$

restricting fiber wise to the cup product pairing and the Hodge filtration on $H^2(X_t, \mathbb{C})$, respectively.

Suppose now, that the local system $\mathcal{H}_{\mathbb{Z}}$ is trivial, and we have chosen a *marking*, i.e. an isometric trivialization $m : \mathcal{H}_{\mathbb{Z}} \to \Lambda \otimes \underline{\mathbb{Z}}_B$. We can transfer the Hodge filtration on $\mathcal{H}_{\mathbb{Z}}$ to the constant system Λ via m and get a unique map to the period domain

$$\mathscr{P}(\mathcal{F}^*, m) : B \longrightarrow \mathscr{D}(\Lambda)$$

with the property that the pull-back of the tautological variation of Hodge structures agrees with $m(\mathcal{F}^*)$ as Hodge structures on $\Lambda \otimes \underline{\mathbb{Z}}_B$. If Ω is a local section of $\mathcal{F}^2$, then the period map is explicitly given by

$$\mathscr{P}(\mathcal{F}^*, m)(t) = [m(\Omega(t))] \in \mathscr{D}(\Lambda)$$

for $t \in B$.

8.8 CFT-Periods of marked complex K3 surfaces

In the same way, we define the periods of the enlarged Hodge structures. We endow the local system

$$\tilde{\mathcal{H}}_{\mathbb{Z}} := \mathbb{R}^* \pi_* \underline{\mathbb{Z}}_X = \mathbb{R}^0 \pi_* \underline{\mathbb{Z}}_X \oplus \mathbb{R}^2 \pi_* \underline{\mathbb{Z}}_X \oplus \mathbb{R}^4 \pi_* \underline{\mathbb{Z}}_X$$

with the Mukai pairing defined by the same formula (8.1) as above. The associated holomorphic vector bundle $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_{\mathbb{Z}} \otimes \mathcal{O}_B$ carries the *B-model Hodge filtration*

$$\mathcal{F}^2_B := \pi_* \Omega^2_{X/B} \subset \mathcal{F}^1_B := (\mathcal{F}^2_B)^{\perp} \subset \tilde{\mathcal{H}}.$$

For every marking $\tilde{m} : \tilde{\mathcal{H}}_{\mathbb{Z}} \to \tilde{\Lambda} \otimes \underline{\mathbb{Z}}_B$ of this enlarged local system, we get an associated *B-model period map*

$$\mathscr{P}_B(\mathcal{F}^*_B, \tilde{m}) : B \longrightarrow \mathscr{D}(\tilde{\Lambda}).$$

Remark 8.7. A marking m of $\mathcal{H}_{\mathbb{Z}}$ determines a marking of $\tilde{\mathcal{H}}_{\mathbb{Z}}$ by the following convention. There are canonical trivializing sections $1 \in \mathbb{R}^0 \pi_* \underline{\mathbb{Z}}_X$ and $or \in \mathbb{R}^4 \pi_* \underline{\mathbb{Z}}_X$, satisfying $(1.or) = -1$ with respect to the Mukai pairing. Let e, f be the standard basis of U with intersections $(e.f) = 1, (e.e) = (f.f) = 0$. Then the map

$$m_0 : \mathbb{R}^0 \pi_* \underline{\mathbb{Z}}_X \oplus \mathbb{R}^4 \pi_* \underline{\mathbb{Z}}_X \longrightarrow U \otimes \underline{\mathbb{Z}}_B, \quad 1 \mapsto e,\ or \mapsto -f$$

is an orthogonal isomorphism of local systems and the map

$$\tilde{m} := m \oplus m_0 : \tilde{\mathcal{H}}_{\mathbb{Z}} = \mathcal{H}_{\mathbb{Z}} \oplus (\mathbb{R}^0 \pi_* \underline{\mathbb{Z}}_X \oplus \mathbb{R}^4 \pi_* \underline{\mathbb{Z}}_X) \longrightarrow (\Lambda \oplus U) \otimes \underline{\mathbb{Z}}_B = \tilde{\Lambda} \otimes \underline{\mathbb{Z}}_B.$$

defines a marking of $\tilde{\mathcal{H}}_{\mathbb{Z}}$.

8.9 CFT-Periods of marked symplectic K3 surfaces

Let $\pi : X \to B$ be a family of K3 surfaces, and $\omega \in H^0(B, \pi_* \mathcal{A}^2_{X/B})$ a $d_{X/B}$-closed two-form, that restricts to a Kähler form on each fiber X_t. The form ω determines a global section of

$$\mathcal{H}_\infty = (\mathbb{R}^2 \pi_* \underline{\mathbb{Z}}_X) \otimes \mathscr{C}^\infty_B(\mathbb{C}) = \mathbb{R}^2 \pi_* (\mathcal{A}^*_{X/B}) = \mathcal{H}^2(\pi_* \mathcal{A}^*_{X/B}).$$

Analogously, a closed form $\beta \in H^0(B, \pi_* \mathcal{A}^2_{X/B})$ gives a section $\beta \in H^0(B, \mathcal{H}_\infty)$. Given ω and β we define a section

$$z = \exp(i\omega + \beta) \in H^0(B, \tilde{\mathcal{H}}_\infty), \quad \tilde{\mathcal{H}}_\infty = \mathbb{R}^* \pi_* \mathbb{Z} \otimes \mathscr{C}^\infty_B(\mathbb{C})$$

by the same formula (8.2) used in the point-wise definition of z. We set the *A-model Hodge filtration* to be the sequence of $\mathscr{C}^\infty$-vector bundles

$$\mathcal{F}^2_A := \mathscr{C}^\infty_B(\mathbb{C})\, z \subset \mathcal{F}^1_A := (\mathcal{F}^2_A)^\perp \subset \tilde{\mathcal{H}}_\infty.$$

In the same way as above, every marking $\tilde{m} : \tilde{\mathcal{H}}_\mathbb{Z} \to \tilde{\Lambda} \otimes \underline{\mathbb{Z}}_B$ determines an *A-model period map*

$$\mathscr{P}_A(\mathcal{F}^*_A, \tilde{m}) : B \longrightarrow \mathscr{D}(\tilde{\Lambda})$$

which is a morphism of $\mathscr{C}^\infty$-manifolds.

Example 8.8. Given d_X-closed two-forms $\omega, \beta \in \mathcal{A}^2_X$ on X, we get $d_{X/B}$-closed relative two-forms via the canonical projection $\mathcal{A}^*_X \to \mathcal{A}^*_{X/B}$. In this case, the map $B \ni t \mapsto \exp(i\omega(t) + \beta(t)) \in \tilde{H}(X_t, \mathbb{C})$ factors through the pull-back

$$i^* : H^*(X, \mathbb{C}) \longrightarrow H^*(X_t, \mathbb{C})$$

along the inclusion $i : X_t \to X$. As this map is already defined on $H^2(\_, \mathbb{Z})$ the associated period map is constant.

We can extend this example a bit further. Let ω be a constant Kähler form as above and $f : B \to \mathbb{H}$ a holomorphic function to the upper half-plane. The form $f\omega = iIm(f)\omega + Re(f)\omega$ is $d_{X/B}$-closed and satisfies $(\exp(f\omega).\overline{\exp(f\omega)}) > 0$. Hence we get a period map

$$\mathscr{P}_A(\mathcal{F}^*_A, \tilde{m})(t) = [\tilde{m}(\exp(f(t)\omega))] \in \mathscr{D}(\tilde{\Lambda})$$

for $t \in B$, which is easily seen to be holomorphic.

8.10 Mirror symmetry for families

Let $\pi : X \to B$ be a family of complex K3 surfaces with marking and $\rho : Y \to C$ a family of K3 surfaces with marking and chosen relative complexified Kähler form $i\omega + \beta$.

Definition 8.9. A *mirror symmetry* between X and Y consists of an orthogonal transformation $g \in O(\tilde{\Lambda})$ called *global mirror map* and an étale, surjective morphism $\psi : C \to B$ called *geometric mirror map*[1] such that the following diagram is commutative.

[1] We think of ψ as a multi-valued isomorphism: In practice the period map for X is only well defined after base-change to a covering space $\tilde{B} \to B$. Moreover, ψ induces an isomorphism between the universal covering spaces of C and B.

$$\begin{tikzcd}
C \arrow[r, "\mathscr{P}_A(Y)"] \arrow[d, "\psi"'] & \mathscr{D}(\tilde{\Lambda}) \arrow[d, "g"] \\
B \arrow[r, "\mathscr{P}_B(X)"] & \mathscr{D}(\tilde{\Lambda})
\end{tikzcd}$$

In particular for every point $s \in C$ we have a mirror pair

$$H_A(Y_s, \mathbb{Z}) \cong H_B(X_{\psi(s)}, \mathbb{Z}).$$

Remark 8.10. A typical global mirror map will exchange the hyperbolic plane $H^0(X_s, \mathbb{Z}) \oplus H^4(X_s, \mathbb{Z})$ with a hyperbolic plane inside $H^2(X_s, \mathbb{Z})$ as in section 8.5. We will see, that this happens in our case, too. Examples for other mirror maps can be found in [Huy04, Sec. 6.4].

Note that, if the markings of X and Y are both induced by a marking of the second cohomology local system as in Remark 8.7 then g can never be the identity. Indeed, we always have

$$H_B^{2,0}(X_s) \perp (H^0(X_s, \mathbb{Z}) \oplus H^4(X_s, \mathbb{Z}))$$

but never $H_A^{2,0}(Y_t) \perp (H^0(Y_t, \mathbb{Z}) \oplus H^4(Y_t, \mathbb{Z}))$ since $(\exp(i\omega + \beta).or) = -1$.

Chapter 9

Period map for the quartic

Since the calculation of the period map for the symplectic quartic is much easier than for the Dwork family, we begin with this construction.

A smooth quartic in $Y \subset \mathbb{P}^3$ inherits a symplectic structure from $\mathbb{P}^3$ by restricting the Fubini–Study Kähler form ω_{FS}. A classical result of Moser [Mos65] shows that all quartics are symplectomorphic.

Proposition 9.1. *For all primitive $h \in \Lambda$ with $\langle h, h \rangle = 4$ there exists a marking $m : H^2(Y, \mathbb{Z}) \to \Lambda$ such that $m([\omega]) = h$.*

Proof. Recall that $[\omega] = [\omega_{FS}|_Y] \in H^2(Y, \mathbb{R})$ is an integral class and satisfies $\int_Y \omega^2 = 4$. Moreover $[\omega]$ is primitive since there is an integral class l, represented by a line on Y, with $l.[\omega] = 1$. Let $n : H^2(Y, \mathbb{Z}) \to \Lambda$ be an arbitrary marking. We can apply a theorem of Nikulin, which we state in full generality below (10.12), to get an isometry of $H^2(Y, \mathbb{Z})$ that maps $[\omega]$ to the primitive vector $n(h)$ of square 4. $\square$

Fix a quartic Y with symplectic form ω. Scaling the symplectic form by $\lambda \in \mathbb{R}_{>0}$ and introducing a B-field $\beta = \mu\,\omega \in H^2(X, \mathbb{R}), \mu \in \mathbb{R}$. We get a family of complexified symplectic manifolds

$$\rho : \mathcal{Y} \longrightarrow \mathbb{H}$$

with fiber $(Y, z = \exp(p\omega))$ over a point $p = i\lambda + \mu \in \mathbb{H}$.

Since the family is topologically trivial, the marking m of Y constructed above extends to a marking of $\rho : \mathcal{Y} \to \mathbb{H}$, which induces an enlarged marking

$$\tilde{m} : R^*\rho_*\underline{\mathbb{Z}}_{\mathcal{Y}} \longrightarrow \underline{\mathbb{Z}}_{\mathbb{H}} \otimes \tilde{\Lambda}.$$

by the procedure explained in Remark 8.7.

Proposition 9.2. *The A-model period map of the family $\rho : \mathcal{Y} \to \mathbb{H}$*

$$\mathscr{P}_A(\mathcal{F}_A^*, \tilde{m}) : \mathbb{H} \longrightarrow \mathscr{D}(\tilde{\Lambda})$$

is holomorphic and induces an isomorphism of $\mathbb{H}$ onto a connected component $\mathscr{D}(\langle h \rangle \oplus U)^+$ of

$$\mathscr{D}(\langle h \rangle \oplus U) \subset \mathscr{D}(\Lambda \oplus U) = \mathscr{D}(\tilde{\Lambda}).$$

Proof. By Example 8.8 the period map is holomorphic. If (h, e, f) is the standard basis of $\langle h \rangle \oplus U$, then it is explicitly given by

$$p \mapsto [\exp(ph)] = [e + ph - \frac{1}{2}p^2(h.h)f] \in \mathscr{D}(\langle h \rangle \oplus U) \subset \mathscr{D}(\tilde{\Lambda}).$$

The injectivity of the period map is now obvious. To show surjectivity we let $[ae + bh + cf]$ be an arbitrary point in $\mathscr{D}(\langle h \rangle \oplus U)$. By definition we have

$$ac + 2b^2 = 0, \quad a\bar{c} + c\bar{a} + 4b\bar{b} = 2Re(a\bar{c}) + 4|b|^2 > 0$$

Hence $a \neq 0$ and we can set $p := b/a$. Then $c/a = -2p^2$, so that

$$[\exp(ph)] = [1e + b/ah + c/af] = [ae + bh + cf].$$

The inequality translates into $Im(p)^2 > 0$. That means

$$\mathbb{C} \setminus \mathbb{R} \longrightarrow \mathscr{D}(\langle h \rangle \oplus U), \ p \mapsto [\exp(ph)]$$

is an isomorphism and therefore proves the proposition. $\square$

Chapter 10

Period map for the Dwork family

10.1 Construction of the Dwork family

We start with the *Fermat pencil* $F \subset \mathbb{P}^3 \times \mathbb{P}^1$ defined by the equation

$$f = X_0^4 + X_1^4 + X_2^4 + X_3^4 - 4tX_0X_1X_2X_3$$

where $X_0, \ldots, X_3$ are homogeneous coordinates on $\mathbb{P}^3$ and $t \in \mathbb{A}^1 \subset \mathbb{P}^1$ is an affine parameter. We view F as a family of quartics over $\mathbb{P}^1$ via the projection $p : F \to \mathbb{P}^1$.

The fibers $F_t = p^{-1}(\{t\})$ are smooth if t does not lie in

$$\Sigma = \{t \,|\, t^4 = 1\} \cup \{\infty\}.$$

For $t^4 = 1$ we find 16 singularities of type A_1, for $t = \infty$ the Fermat pencil degenerates into the union of four planes: $X_0X_1X_2X_3 = 0$.

Let μ_4 denote the forth roots of unity. The group

$$G = \{\, (a_0, a_1, a_2, a_3) \mid a_i \in \mu_4, a_0a_1a_2a_3 = 1 \,\} / \mu_4 \cong (\mathbb{Z}/4\mathbb{Z})^2$$

acts on F respecting the fibers F_t.

The quotient variety $S = F/G$ can be explicitly embedded into a projective space as follows. The monomials

$$(Y_0, \ldots, Y_4) := (X_0^4, X_1^4, X_2^4, X_3^4, X_0X_1X_2X_3)$$

define a G-invariant map $\mathbb{P}^3 \to \mathbb{P}^4$, and the image of F in $\mathbb{P}^4 \times \mathbb{P}^1$ under this morphism is cut out by the equations

$$Y_0 + Y_1 + Y_2 + Y_3 - 4tY_4, \quad Y_0Y_1Y_2Y_3 - Y_4^4. \tag{10.1}$$

It is easy to see that this image is isomorphic to the quotient S.

Proposition 10.1. *For $t \notin \Sigma$ the space S_t has precisely six singularities of type A_3. If $t^4 = 1$ there is an additional A_1-singularity. The fiber S_∞ is a union of hyperplanes, it is in fact isomorphic to F_∞ itself.*

Proof. The first statement can be seen by direct calculation using (10.1). A more conceptual argument goes as follows. We note that the action of G is free away from the 24 points in

$$\cup_{i\neq j}\,\{\,[X_0:\cdots:X_3]\mid X_i=X_j=0, F=0\,\}$$

which have stabilizer isomorphic to μ_4. Around such a point p we find an analytic neighborhood U such that the stabilizer G_p acts on U and S_t is locally isomorphic to U/G_p.

$$\begin{tikzcd} F_t \arrow[d, "\phi"'] & U \arrow[l, hook'] \arrow[d] & \ni p \\ S_t & U/G_p \arrow[l, hook'] \end{tikzcd}$$

We can choose $U\subset\mathbb{C}^2$ to be a ball on which $G_p\cong\mu_4$ acts as

$$a\cdot(x,y)=(ax,a^{-1}y)$$

The quotient singularity is well known to be of type $D_2=A_3$.

To prove the second statement, recall that there are 16 singularities of Type A_1 in each surface F_t for $t^4=1$. It is easy to see that these form an orbit for the G action and that they are disjoint from the A_3-singularities above.

Finally, that S_∞ is a union of hyperplanes follows directly form the equations (10.1). □

Note that $S_t\subset\mathbb{P}^4$ is isomorphic to a (singular) quartic in $\mathbb{P}^3$ since the first equation defining S_t is linear.

Proposition 10.2. *There exists a minimal, simultaneous resolution of the A_3 singularities in $S\to\mathbb{P}^1$. That means, there is a threefold $X\to\mathbb{P}^1$ together with a morphism $X\to S$ over $\mathbb{P}^1$ which restricts to a minimal resolution of the six A_3-singularities on each fiber over $t\notin\Sigma$.*

Proof. The position of the A_3-singularities of S_t in $\mathbb{P}^4$ does not change, as we vary t. So we can blow-up $\mathbb{P}^4$ at these points. Also the singularities of the strict transform of S_t are independent of t. Hence we can construct X by blowing-up the singularities again. □

Definition 10.3. The family $X\to\mathbb{P}^1$ is called the the *Dwork Family.*

The fibers X_t are smooth for $t\in B=\mathbb{P}^1\setminus\Sigma$, $\Sigma=\{t\,|\,t^4=1\}\cup\infty$. We denote by $\pi: X\to B$ the restriction.

Proposition 10.4. *The members X_t of the Dwork family are $K3$ surfaces for $t\notin\Sigma$.*

Proof. It is shown in [Nik76], that a minimal resolution of a quotient of a K3 surface by a finite group acting symplectically is again a K3 surface. □

10.2 Holomorphic two-forms on the Dwork family

In this section we construct holomorphic two-forms Ω_t on the members of the Dwork family. We do this first for the Fermat pencil using the residue construction ([CMSP03] Section 3.3, [GH78] Chapter 5) and then pull back to the Dwork family.

Let $U := \mathbb{P}^3 \setminus F_t$, there is a residue morphism:

$$Res : H^k(U, \mathbb{C}) \to H^{k-1}(F_t, \mathbb{C}).$$

This morphism is most easily described for de Rham cohomology groups. The boundary of a tubular neighborhood of F_t in $\mathbb{P}^3$ will be a S^1-bundle over F_t completely contained in U. We integrate a k-form on U fiber-wise along this bundle to obtain a $k-1$ form on F_t, this induces the residue map in cohomology.

Remark 10.5. The residue morphism is also defined on the integral cohomology groups. It is the composition of the boundary morphism in the long exact sequence of the space pair $(\mathbb{P}^3, U)$ with the Thom isomorphism $H^{k+1}(\mathbb{P}^3, U) \cong H^{k-1}(F_t)$ (up to a sign).

There is a unique (up to scalar) holomorphic 3-form Ξ_t on $\mathbb{P}^3$, with simple poles along F_t. Its pull-back to $\mathbb{C}^4 \setminus \{0\}$ is given by the expression

$$\Xi_t = \sum_{i=0}^{3} (-1)^i \frac{X_i \, dX_0 \wedge \cdots \wedge \widehat{dX_i} \wedge \cdots \wedge dX_3}{X_0^4 + X_1^4 + X_2^4 + X_3^4 - 4tX_0X_1X_2X_3}. \tag{10.2}$$

One checks that this form is closed and hence $\sigma_t := Res(\Xi_t)$ is a well defined, closed two-form on F_t.

Let us choose coordinates $z_i = X_1/X_0, i = 1, \dots, 3$ for $\mathbb{P}^3$, here

$$\sigma_t = Res(\Xi_t) = Res(\frac{dz_1 \wedge \cdots \wedge dz_3}{f_t})$$

where $f_t = 1 + z_1^4 + z_2^4 + z_3^4 - 4tz_1z_2z_3$ is the function defining F_t.

On the open subset $\partial f_t / \partial z_3 \neq 0$ the functions (z_1, z_2) are (étale) coordinates for F_t, and (f, z_1, z_2) are (étale) coordinates for $\mathbb{P}^3$. In these coordinates the sphere bundle is just given by $|f| = \varepsilon > 0$ and fiber-wise integration reduces to taking the usual residue in each fiber $(z_1, z_2) = const.$

Solving $df = \sum_i \partial f / \partial z_i dz_i$ for dz_3 and substituting above we get a local coordinate expression for σ_t:

$$\sigma_t = Res(\frac{df}{f} \cdot \frac{dz^1 \wedge dz^2}{\partial f / \partial z_3}) = 2\pi i \frac{dz_1 \wedge dz_2}{4z_3^3 - 4tz_1z_2}.$$

Proposition 10.6. *The residue σ_t of the meromorphic three form Ξ_t, is a nowhere-vanishing holomorphic two form on all smooth members F_t of the Fermat pencil.* □

The same construction gives us a global version of σ_t: The inclusion $F \subset \mathbb{P}^3 \times B$ is a smooth divisor, and the residue of the three-form Ξ on $\mathbb{P}^3 \times B$ given by same formula (10.2) provides us with a two-form σ on F which defines a global section of $p_*\Omega^2_{F/B}$. Clearly σ restricts to σ_t on each fiber and hence trivializes the line bundle $p_*\Omega^2_{F/B}$.

We now proceed to the Dwork family. Consider the group

$$G = \{\, (a_0, a_1, a_2, a_3) \mid a_i \in \mu_4, a_0a_1a_2a_3 = 1 \,\} / \mu_4$$

acting on $F_t \subset \mathbb{P}^3$. For $g = (a_0, a_1, a_2, a_3) \in G$ we compute

$$g^*\Xi_t = \sum_{i=0}^{3} (-1)^i \frac{a_0 \dots a_3 X_i \, X_0 \wedge \dots \wedge \widehat{dX_i} \wedge \dots \wedge dX_3}{a_0^4X_0^4 + a_1^4X_1^4 + a_2^4X_2^4 + a_3^4X_3^4 - 4ta_0 \dots a_3X_0X_1X_2X_3}$$

which equals Ξ_t, hence $\sigma_t = Res(\Xi_t)$ is also G invariant. It follows that σ_t descends to a form $\tilde{\sigma}$ on the smooth part $S_t^{reg} \subset S_t = F_t/G$.

Recall that the Dwork family is a simultaneous, minimal resolution of singularities $\rho : X_t \to S_t$. In particular ρ is an isomorphism over S_t^{reg}.

As S_t^{reg} is isomorphic to an open subset of a K3 surface we find $\Omega^2_{S_t^{reg}} \cong \mathcal{O}_{S_t^{reg}}$. Moreover $H^0(S_t^{reg}, \mathcal{O}_{S_t^{reg}}) = \mathbb{C}$ since the complement is an exceptional divisor. It follows, that $\tilde{\sigma}$ extends to a holomorphic 2-form Ω_t on X_t.

The same construction works also in the global situation $F \to S \leftarrow X$ over B and gives us a global section Ω of $\pi_*\Omega^2_{X/B}$.

Proposition 10.7. *There is a global section Ω of $\pi_*\Omega^2_{X/B}$ that restricts to Ω_t on each fiber. Moreover the pull-back of Ω_t along the rational map $F_t \dashrightarrow X_t$ coincides with σ_t on the set of definition.*

The section Ω trivializes the line bundle $\pi_*\Omega^2_{X/B}$ and thus the variation of Hodge structures of $\pi : X \to B$ is given by

$$\mathcal{F}^2 = \mathcal{O}_B \, \Omega \;\subset\; \mathcal{F}^1 = (\mathcal{F}^2)^\perp \;\subset\; \mathcal{H}$$

10.3 Monodromy of the Dwork family

The Dwork family $\pi : X \to B$ determines a local system $\mathcal{H}_\mathbb{Z} := \mathbb{R}^2\pi_*\mathbb{Z}_X$ on B. As is well known, every local system is completely determined by its monodromy representation

$$PT : \pi_1(B, t) \longrightarrow Aut((\mathcal{H}_\mathbb{Z})_t) = Aut(H^2(X_t, \mathbb{Z})), \quad t \in B$$

given by parallel transport. In this chapter we will explicitly describe this representation.

To state the main result we need the following notation. Let

$$M_2 = 2E_8(-1) \oplus U \oplus \langle -4 \rangle \quad \text{and} \quad T_0 = \langle 4 \rangle \oplus U.$$

If e, f is the standard basis of U, and l, h are generators of $\langle -4 \rangle$ and $\langle 4 \rangle$ respectively then we can define a primitive embedding

$$\langle -4 \rangle \oplus \langle 4 \rangle \longrightarrow U, \quad l \mapsto e - 2f,\; h \mapsto e + 2f.$$

This induces also an embedding $M_2 \oplus T_0 \to 2E_8(-1) \oplus 3U = \Lambda$.

Theorem 10.8 (Narumiyah–Shiga, Dolgachev). *At the point $t_0 = i/\sqrt{2}$ there is an isomorphism*

$$m : H^2(X_{t_0}, \mathbb{Z}) \longrightarrow \Lambda$$

such that

i) The Néron–Severi group of each member X_t contains the image of M_2 under m^{-1} composed with parallel transport along any path from t_0 to t in B. For general t this inclusion is an isomorphism.

ii) The monodromy representation on $H^2(X_t, \mathbb{Z})$ respects the images of the subspaces M_2, T_0 and acts trivially on the first one.

Moreover, the monodromy representation on T_0 is given by the following matrices. Let (h, e, f) be the standard basis of $T_0 = \langle 4 \rangle \oplus U$, let $\gamma_k \in \pi_1(B, t_0)$ be the paths depicted in Figure 10.3 and $\gamma_\infty = (\gamma_4 \cdot \gamma_3 \cdot \gamma_2 \cdot \gamma_1)^{-1}$ the path around $\infty \in \mathbb{P}^1$.[1]

Then the following identities hold

$$PT_{\gamma_k}(h, e, f) = (h, e, f).M^k$$

where

$$M^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M^2 = \begin{pmatrix} 5 & 1 & -3 \\ -12 & -2 & 9 \\ 4 & 1 & -2 \end{pmatrix}, \quad M^3 = \begin{pmatrix} 17 & 6 & -6 \\ -24 & -8 & 9 \\ 24 & 9 & -8 \end{pmatrix}$$

$$M^4 = \begin{pmatrix} 5 & 3 & -1 \\ -4 & -2 & 1 \\ 12 & 9 & -2 \end{pmatrix}, \quad M^\infty = \begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ -16 & -32 & 1 \end{pmatrix}.$$

Remark 10.9. Note that the matrix M^∞ is unipotent of maximal index 3, i.e.

$$(M^\infty - 1)^3 = 0, \quad (M^\infty - 1)^2 \neq 0$$

this will be crucial for the characterization of the period map in chapter 10.6.

The proof is a consequence of the following theorems.

Theorem 10.10 (Dolgachev [Dol96]). *The Dwork family $X \to B$ carries an M_2-polarization, i.e. there exists a morphism of local systems*

$$Pol : M_2 \otimes \underline{\mathbb{Z}}_B \longrightarrow \mathcal{H}_\mathbb{Z} \tag{10.3}$$

inducing a primitive lattice embedding in each fiber which factorizes through the inclusion $\mathrm{Pic}(X_t) \subset H^2(X_t, \mathbb{Z})$*. Moreover, for general t this map is an isomorphism onto the Neron–Severi group.*

Theorem 10.11 (Narumiyah–Shiga [NS01]). *There is a primitive lattice embedding*

$$Tr : T_0 \longrightarrow H^2(X_{t_0}, \mathbb{Z})$$

with image in the orthogonal complement of the polarization $Pol(M_2)_{t_0}$. Moreover the monodromy representation on T_0 is given by the matrices described in Theorem (10.8).

[1]As Narumiyah and Shiga, we use the convention to compose paths like functions, i.e. $\gamma : p \to q$, $\delta : q \to r$, then $\delta \cdot \gamma : p \to r$. This has the advantage, that monodromy becomes a representation, as opposed to an anti-representation.

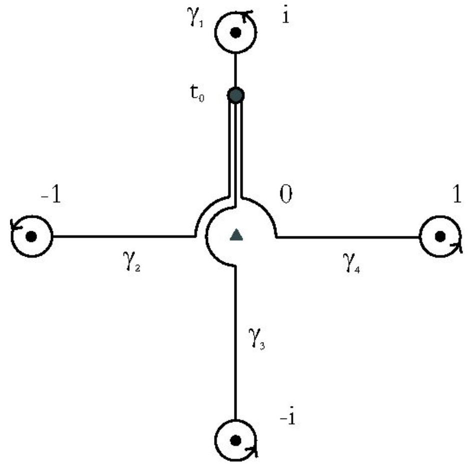


Figure 10.1: Generators of the fundamental group $\pi_1(B, t_0)$.

Proof. The intersection form is stated in Theorem 4.1 of [NS01], and the monodromy matrices in Remark 4.2 following this theorem. We only explain how their notation differs form ours.

They consider the family $\tilde{F}_\lambda \subset \mathbb{P}^3$ defined by the equation

$$X_0^4 + X_1^4 + X_2^4 + X_3^4 + \lambda X_0 X_1 X_2 X_3 = 0.$$

In order to ensure the relation $\lambda = 4t$ holds, we identify this family via the isomorphism

$$\tilde{F}_\lambda \to F_t, \quad X_0 \mapsto -X_0,\ X_1 \mapsto X_1,\ X_2 \mapsto X_2,\ X_3 \mapsto X_3$$

with our Fermat pencil.

Their basis (e', f', h') of $U \oplus \langle 4 \rangle$ is related to our basis (h, e, f) of $\langle 4 \rangle \oplus U$ by

$$(h, e, f) = (e', f', h').T, \quad T := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

They introduce a new variable $t' = -(\lambda)^2/2$ and consider paths $\delta_1, \ldots, \delta_3$ in the t'-plane (Fig.6 in [NS01]). The images of our paths $\gamma_1, \ldots, \gamma_4$ are give by

$$\gamma_1 \mapsto \delta_1, \quad \gamma_2 \mapsto \delta_2^{-1} \cdot \delta_3 \cdot \delta_2, \quad \gamma_3 \mapsto \delta_2^{-1} \cdot \delta_1 \cdot \delta_2, \quad \gamma_4 \mapsto \delta_3.$$

Let N_i be the monodromy matrices along δ_i as stated in Remark 4.2 of [NS01]. By what was said above, we compute the monodromy matrix e.g. along γ_2 as

$$M^2 = T^{-1}.(N_2)^{-1}.N_3.(N_2).T. \qquad \square$$

So far we do not know whether the primitive embedding

$$Pol_{t_0} \oplus Tr : M_2 \oplus T_0 \longrightarrow H^2(X_{t_0}, \mathbb{Z})$$

can be extended to an isomorphism of lattices $\Lambda \longrightarrow H^2(X_{t_0}, \mathbb{Z})$.

The following theorem of Nikulin ensures, that we can always change Pol_{t_0} by an automorphism of M_2 such, that an extension exists.

Theorem 10.12 (Nikulin [Nik79], 1.14.4)**.** *Let $i : S \to L$ be a primitive embedding of an even non-degenerate lattice S of signature (s_+, s_-) into an even non-degenerate lattice of signature (l_+, l_-). For any other primitive embedding $j : S \to L$, there is an automorphism $\alpha \in O(L)$ such that $i = j \circ \alpha$ if*

$$l_+ > s_+, l_- > s_- \quad \textit{and} \quad rk(L) - rk(S) \geq l(S) + 2$$

where $l(S)$ is the minimal number of generators of the discriminant group $S^\vee/S$.

We apply this theorem as follows. First choose an arbitrary isomorphism $\tilde{n} : \Lambda \to H^2(X_{t_0}, \mathbb{Z})$. This gives us a primitive embedding of T_0 by restriction

$$\tilde{n}|_{T_0} : T_0 \longrightarrow H^2(X_{t_0}, \mathbb{Z}).$$

Also there is the primitive embedding constructed in (10.11)

$$Tr : T_0 \longrightarrow H^2(X_{t_0}, \mathbb{Z}).$$

Note that $sign(T_0) = (2, 1), sign(\Lambda) = (3, 19)$ and $l(T_0) = l(\langle 4 \rangle) = 1$, so we can apply Nikulin's theorem to conclude, that these two differ by an orthogonal automorphism α of $H^2(X_{t_0}, \mathbb{Z})$.

Set $n = \alpha \circ \tilde{n}$ so that $\tilde{n}|_{T_0} = Tr$. Note also that n induces an isomorphism of the orthogonal complements

$$n|_{M_2} : M_2 = T_0^\perp \longrightarrow Tr(T_0)^\perp = Pol(M_2)_{t_0}.$$

As mentioned above, this isomorphism can differ, by an automorphism of M_2, from the one provided by Dolgachev's polarization. It is now clear that $m = n^{-1}$ is a marking with the required properties. This concludes the proof of Theorem 10.8. $\square$

Corollary 10.13. *The local system $\mathcal{H}_\mathbb{Q} := \mathcal{H}_\mathbb{Z} \otimes \mathbb{Q}$ decomposes into an orthogonal direct sum*

$$\mathcal{H}_\mathbb{Q} = \mathcal{P}_\mathbb{Q} \oplus \mathcal{T}_\mathbb{Q}$$

where $\mathcal{P}_\mathbb{Q}$ is a trivial local system of rank 19 spanned by the algebraic classes in the image of the polarization Pol, and $\mathcal{T}_\mathbb{Q}$ is spanned by the image of Tr.

10.4 The Picard–Fuchs equation

So far we have described the local system $\mathcal{H}_\mathbb{Z}$ and the Hodge filtration $\mathcal{F}^i \subset \mathcal{H}$ of the Dwork family independently. The next step is to relate them to each other by calculating the period integrals

$$t \mapsto \int_\Gamma \Omega_t$$

for local sections $\Gamma \in \mathcal{H}_\mathbb{Z}$. The essential tool here is a differential equation, the Picard–Fuchs equation, that is satisfied by these period integrals.

Let t be the affine coordinate on $B \subset \mathbb{A}^1$, and ∂_t the associated global vector field. The *Gauß–Manin connection* ∇ on $\mathcal{H} = \mathcal{H}_\mathbb{Z} \otimes \mathcal{O}_B$ is defined by $\Gamma \otimes f \mapsto \Gamma \otimes df$. We denote by

$$\Omega^{(i)} := \nabla_{\partial_t} \circ \cdots \circ \nabla_{\partial_t} \Omega \ \in \ H^0(B, \mathcal{H})$$

be the i-th iterated Gauß–Manin derivative of Ω in direction ∂_t.

Proposition 10.14. *The global section Ω of $\pi_*\Omega^2_{X/B}$ satisfies the differential equation*

$$\Omega^{(3)} = \frac{1}{1-t^4}(6t^3\Omega^{(2)} + 7t^2\Omega^{(1)} + t\Omega). \tag{10.4}$$

Proof. This is an application of the Griffiths–Dwork reduction method, see [Gri69], or [Mor92] for a similar application. We will outline the basic steps.

It is enough to prove the formula on the dense open subset $\rho(S_t^{reg})$ of X. Since the map $F \to S$ is étale over S_t^{reg}, we can furthermore reduce the calculation to the Fermat pencil of quartic hypersurfaces. The holomorphic forms on the members F_t of the Fermat pencil are residues of meromorphic 3-forms Ξ_t on $\mathbb{P}^3$. Since taking residues commutes with the Gauß–Manin connection, we only need to differentiate the global 3-from Ξ_t.

We then use a criterion of Griffiths to show the corresponding equality between the residues. This involves a Gröbner basis computation in the Jacobi ring of F_t. See e.g. [Smi07] for an implementation. □

Definition 10.15. We define the *Picard–Fuchs operator* associated to the Dwork family $X \to B$ to be the differential operator

$$\mathcal{D} = \partial_t^3 - \frac{1}{1-t^4}(6t^3\partial_t^2 + 7t^2\partial_t + t) \tag{10.5}$$

obtained from (10.4) by replacing ∇ with ∂_t.[2]

Remark 10.16. Let $\Gamma_t \in H^2(X_t, \mathbb{Z})$ be a cohomology class. Extend Γ_t to a flat local section Γ of $\mathcal{H}_\mathbb{Z}$. Since the quadratic form $(\,.\,)$ on $\mathcal{H}_\mathbb{Z}$ is also flat, we can calculate

$$\partial_t\langle\Gamma, \Omega\rangle = \langle\Gamma, \nabla_{\partial_t}\Omega\rangle = \langle\Gamma, \Omega^{(1)}\rangle.$$

A similar calculation shows, that the function

$$t \mapsto \int_{\Gamma_t} \Omega_t = \langle\Gamma, \Omega\rangle(t)$$

is a solution of the Picard–Fuchs equation $\mathcal{D} = 0$.

10.5 The period map of the Dwork family

Recall that the Dwork family

$$\pi : X \longrightarrow B, \quad B = \mathbb{P}^1 \setminus \Sigma, \quad \Sigma = \{t \,|\, t^4 = 1\} \cup \infty$$

determines a variation of Hodge structures on B:

$$\mathcal{H}_\mathbb{Z} := R^2\pi_*\underline{\mathbb{Z}}_X, \quad \mathcal{F}^2 = \pi_*\Omega^2_{X/B} \;\subset\; \mathcal{F}^1 = (\mathcal{F}^2)^\perp \;\subset\; \mathcal{H} := \mathcal{H}_\mathbb{Z} \otimes \mathcal{O}_B.$$

We let $c : \tilde{B} \to B$ be the universal cover, and choose a point $\tilde{t_0} \in \tilde{B}$ mapping to $t_0 = i/\sqrt{2}$.

[2] See [Mor92] or [Pet86] for a more general definition of the Picard–Fuchs equation.

Proposition 10.17. *The isomorphism constructed in Theorem 10.8*

$$m : H^2(X_{t_0}, \mathbb{Z}) \longrightarrow \Lambda$$

induces a marking of the local system $c^*\mathcal{H}_{\mathbb{Z}}$.

Proof. We compose m with the canonical isomorphisms

$$(c^*\mathcal{H}_{\mathbb{Z}})_{\tilde{t_0}} \longrightarrow (\mathcal{H}_{\mathbb{Z}})_{t_0} \longrightarrow H^2(X_{t_0}, \mathbb{Z})$$

and extend this map by parallel transport to an isomorphism of local systems

$$m : c^*\mathcal{H}_{\mathbb{Z}} \longrightarrow \mathbb{Z}_{\tilde{B}} \otimes \Lambda.$$

This is possible since $\tilde{B}$ is simply connected, and hence both local systems are trivial. $\square$

Choosing the marking in this way we get a period map

$$\mathscr{P} := \mathscr{P}(c^*\mathcal{F}^*, m) : \tilde{B} \longrightarrow \mathscr{D}(\Lambda).$$

Proposition 10.18. *Let $M_2, T_0 \subset \Lambda$ be as in Theorem 10.8. The period map takes values in $\mathscr{D}(T_0) \subset \mathscr{D}(\Lambda)$.*

Proof. Let $D \in \mathcal{H}_{\mathbb{Z}}$ be a local section contained in the orthogonal complement $m^{-1}(M_2)$ of $m^{-1}(T_0)$. By Dolgachev's theorem 10.10, D is fiber-wise contained in the Picard group, hence $(D.\Omega) = 0$ by orthogonality of the Hodge decomposition. $\square$

Let (h, e, f) be the standard basis of $T_0 = \langle 4 \rangle \oplus U$, we denote by the same symbols also the global sections of $c^*\mathcal{H}_{\mathbb{Z}}$ associated via the marking. By the last proposition we find holomorphic functions a, b, c on $\tilde{B}$ such that

$$c^*\Omega = a\,h + b\,e + c\,f \;\in\; H^0(\tilde{B}, c^*\mathcal{H}) \tag{10.6}$$

and hence

$$\mathscr{P} = [a : b : c] : \tilde{B} \longrightarrow \mathbb{P}(T_0) \subset \mathbb{P}(\Lambda \otimes \mathbb{C}),$$

using the abusive notation $[a : b : c] := [ah + be + cf]$.

Remark 10.19. For each point $\tilde{p} \in \tilde{B},\ p = c(\tilde{p})$ there is a canonical isomorphism of stalks

$$c^* : \mathcal{O}_{B,p} \longrightarrow \mathcal{O}_{\tilde{B},\tilde{p}}, \quad f \mapsto f \circ c.$$

In this way we may view functions on $\tilde{B}$ locally (on $\tilde{B}$) as functions on B.

Proposition 10.20. *If we view the functions a, b, c locally as functions on B, then these functions satisfy the Picard–Fuchs equation (10.5).*

Proof. We can express a, b, c as intersections with the dual basis in the following way. If $(h^\vee, e^\vee, f^\vee) = (h, e, f).G^{-1} = (1/4h, f, e)$, where G is the Gram matrix of $(\,.\,)$ on the basis (h, e, f), i.e.

$$G = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

then $a = (h^\vee.\Omega), b = (e^\vee.\Omega)$ and $c = (f^\vee.\Omega)$. This exhibits the functions a, b, c as period integrals and therefore shows that they satisfy the Picard–Fuchs equation. $\square$

Proposition 10.21. *The germs of the functions a, b, c at $\tilde{p}$ form a basis for the three-dimensional vector space $Sol(\mathcal{D}, p) \subset \mathcal{O}_{B,p}$ of solutions of the Picard–Fuchs equation for all $\tilde{p} \in \tilde{B}$.*

Proof. Linear independence of a, b, c is equivalent to the non-vanishing of the Wronski determinant

$$W = det \begin{pmatrix} a & b & c \\ \partial_t a & \partial_t b & \partial_t c \\ \partial_t^2 a & \partial_t b & \partial_t^2 c \end{pmatrix}$$

of this sections. As the differential equation (10.5) is normalized, this determinant is either identically zero or vanishes nowhere.[3] If the vectors are everywhere linearly dependent, then we get a relation between the Gauß–Manin derivatives $\Omega, \Omega^{(1)}, \Omega^{(2)}$, since

$$\Omega^{(1)} = \nabla_{\partial t}\Omega = (\partial_t a)\, h + (\partial_t b)\, e + (\partial_t c)\, f.$$

This means, that there is a order-two Picard–Fuchs equation for our family. That this is not the case, follows directly from the Griffiths–Dwork reduction process (Proposition 10.14). □

10.6 Characterization of the period map via monodromies

We have seen, that the coefficients of the period map satisfy the Picard–Fuchs equation. In this chapter we characterize these functions among all solutions. The key ingredient is the monodromy calculation in Theorem 10.8.

Remark 10.22. We briefly explain how analytic continuation on B is related to global properties of the function on the universal cover $\tilde{B}$ and thereby introduce some notation.

Let $\tilde{p}$ be a point in $\tilde{B}$, mapping to $p = c(\tilde{p}) \in B$ and let $\delta : p \to q$ be a path in B. There is a unique lift of δ to $\tilde{B}$ starting at $\tilde{p}$. Denote this path by $\tilde{\delta} : \tilde{p} \to \tilde{q}$ and define $\delta \cdot \tilde{p} := \tilde{q}$.

Also we can analytically continue holomorphic functions along δ, this gives us a partially defined morphism between the stalks

$$AC_\delta : \mathcal{O}_{B,p} \longrightarrow \mathcal{O}_{B,q}.$$

A theorem of Cauchy [Inc44] ensures that if a function satisfies a differential equation of the form (10.5), then it can be analytically continued along every path.

These two constructions are related as follows. Let $f : \tilde{B} \to \mathbb{C}$ be a holomorphic function. We can analytically continue the germ $f_{\tilde{p}} \in \mathcal{O}_{B,p}$ along δ and get $AC_\delta f_{\tilde{p}} = f_{\tilde{q}},\ \tilde{q} = \delta \cdot \tilde{p}$.

Suppose now, that δ has the same start and end point $t_0 = i/\sqrt{2} \in B$. We can express the analytic continuation of $\mathscr{P}$ along this paths in terms of the monodromy matrices of $\mathcal{H}_{\mathbb{Z}}$.

[3] A standard reference is [Inc44], but see [Beu07] for a readable summary.

Proposition 10.23. *Let $\delta \in \pi_1(B, t_0)$ and*

$$PT_\delta(h, e, f) = (h, e, f).M^\delta$$

be the monodromy representation of the local system $\mathcal{H}_\mathbb{Z}$ as in Theorem 10.8. The analytic continuation of the period map at $\tilde{t_0}$ is given by

$$AC_\delta \mathscr{P}_{\tilde{t_0}} = AC_\delta[a : b : c] = [a' : b' : c']$$

as tuple of germs at $\tilde{t_0}$, where

$$(a', b', c') = (a, b, c).G.M^\delta.G^{-1}$$

Proof. As remarked above we have the identity of tuples of functions on $\tilde{B}$

$$(a, b, c) = ((h^\vee, e^\vee, f^\vee).\Omega(p)) = (\_.\Omega) \circ (h, e, f).G^{-1}.$$

Now integrals of the form $\int_{\Gamma(\tilde{p})} \Omega(p) = (\Gamma.\Omega)(\tilde{p})$ can be analytically continued by transporting the cycle Γ in the local system. Thus we conclude

$$\begin{aligned} AC_\delta(a, b, c) &= AC_\delta(\_.\Omega) \circ (h, e, f).G^{-1} = (\_.\Omega) \circ (PT_\delta(h, e, f)).G^{-1} \\ &= (\_.\Omega) \circ (h, e, f).M^\delta.G^{-1} = (a, b, c).G.M^\delta.G^{-1}. \end{aligned}$$ □

We already saw in Proposition 9.2 that the period domain $\mathscr{D}(\langle 4\rangle \oplus U) = \mathscr{D}(T_0)$ is isomorphic to $\mathbb{C} \setminus \mathbb{R}$. Let (h, e, f) be the standard basis of $\langle 4\rangle \oplus U$. A slightly different isomorphism is given by

$$e\tilde{x}p : \mathbb{C} \setminus \mathbb{R} \longrightarrow \mathscr{D}(T_0),\ z \mapsto [z\,h\ -\ 1\,e\ +\ 2z^2\,f] \tag{10.7}$$

with inverse $e\tilde{x}p^{-1} : [a\,h\ +\ b\,e\ +\ c\,f] \mapsto -a/b$.

We consider the period map as a function to the complex numbers using this parametrization of the period domain:

$$\mathscr{P}^c = e\tilde{x}p^{-1} \circ \mathscr{P} : \tilde{B} \longrightarrow \mathbb{C}.$$

We will see later, that the period map takes values in the upper half plane.

Theorem 10.8 has a translation into properties of this function.

Proposition 10.24. *The analytic continuation of the germ of the period map at $\tilde{t_0}$ along the paths γ_k depicted in Figure 10.3 is given by*

$$AC_{\gamma_k} \mathscr{P}^c_{\tilde{t_0}} = \beta_k(\mathscr{P}^c_{\tilde{t_0}})$$

where $\beta_k : \mathbb{H} \to \mathbb{H}$ are the Möbius transformations:

$$\beta_1(z) = \frac{-1}{2z},\ \beta_2(z) = \frac{1-2z}{2-6z},\ \beta_3(z) = \frac{3-4z}{4-6z},\ \beta_4(z) = \frac{3-2z}{2-2z},$$
$$\beta_\infty(z) = 4 + z.$$

Proof. Direct calculation using Proposition 10.23. □

The modification (10.7) of the parametrization was introduced to bring the monodromy at infinity to this standard form.

The fixed points of β_i are

$$\beta_1 : \pm i/\sqrt{2},\ \ \beta_2 : \frac{1}{3}(1 \pm i/\sqrt{2}),\ \ \beta_3 : \frac{1}{3}(2 \pm i/\sqrt{2}),\ \ \beta_4 : 1 \pm i/\sqrt{2}. \tag{10.8}$$

These are also the limiting values of the period map at the corresponding boundary points $i, -1, -i, 1 \in \mathbb{P}^1 \setminus B$.

The following characterization of the period map in terms of monodromies is crucial. We show that the period map is determined up to a constant by the monodromy at a maximal unipotent point (cf. Remark 10.9). This is similar to the characterization of the mirror map by Morrison [Mor92, Sec. 2]. The remaining constant can be fixed by considering an additional monodromy transformation.

Proposition 10.25. *Let $a', b' \in \mathcal{O}_{t_0}$ be non-zero solutions to the Picard–Fuchs equation and $\mathscr{P}' := a'/b'$. If*

$$AC_{\gamma_\infty}(a', b') = (a', b').\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix},$$

then there is a $\mu \in \mathbb{C}$ such that $\mathscr{P}' = \mathscr{P}^c + \mu$ as germs at $\tilde{t_0}$.

If furthermore

$$AC_{\gamma_1}\mathscr{P}' = \beta(\mathscr{P}')$$

for a Möbius transformation β with fixed points $\pm i/\sqrt{2}$, then $\mathscr{P}' = \mathscr{P}^c$.

Proof. By Proposition 10.21 the functions a', b' are a $\mathbb{C}$-linear combination of a, b, c. The monodromy transformation of (a, b, c) at infinity is

$$N^\infty := G.M^\infty.G^{-1} = \begin{pmatrix} 1 & 0 & 16 \\ -4 & 1 & -32 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that a, b have the same monodromy behavior as $-a', b'$ at infinity. The matrix N^∞ is unipotent of index 3, i.e. $(N^\infty - \mathrm{id})^3 = 0$, $(N^\infty - \mathrm{id})^2 \neq 0$. In particular the only eigenvalue is 1 and the corresponding eigenspace is one-dimensional, spanned by $e_2 = (0, 1, 0)^t$. Hence there is a $\lambda \in \mathbb{C}$ such that $b' = \lambda b$.

The vector $v = (1, 0, 0)^t$ is characterized by the property $(N^\infty - 1)v = -4e_2$. The space of such v is a one dimensional affine space over the eigenspace $\mathbb{C}\, e_2$. We conclude that $-a' = \lambda a - \mu b$, for some $\mu \in \mathbb{C}$. Since $b \neq 0$ it is $\lambda \neq 0$ and we may assume $\lambda = 1$. Hence

$$\mathscr{P}' = a'/b' = -a/b + \mu = \mathscr{P}^c + \mu.$$

Moreover the monodromy of this function along γ_1 is

$$AC_{\gamma_1}\mathscr{P}' = AC_{\gamma_1}\mathscr{P}^c + \mu = \beta_1(\mathscr{P}^c) + \mu.$$

The fixed point equation $\beta_1(z) + \mu = z$ is a polynomial of degree 2 with discriminant $-2 + \mu^2$. This means the difference of the two solution is $i\sqrt{2}$ only if $\mu = 0$. □

10.7 Nagura and Sugiyama's solutions

Solutions to the Picard–Fuchs equation matching the criterion 10.25 were produced by Nagura and Sugiyama in [NS95]. To state their result, we first need to transform the equation.

The first step is to change the form Ω to $t^{-1}\Omega$, which does not affect the period map, but changes the Picard–Fuchs equation from $\mathcal{D} = 0$ to $\mathcal{D}.t = 0$. We can further multiply by $(1-t^4)$ from the left, without changing the solution space. This differential equation now does descend along the covering map

$$z : B \setminus \{0\} \longrightarrow \mathbb{P}^1 \setminus \{0, 1, \infty\}, \quad t \mapsto z(t) = t^{-4}$$

to a hypergeometric system on C.

Proposition 10.26. *Let*

$$ {}_3\mathcal{D}_2 := \vartheta^3 - z(\vartheta + 1/4)(\vartheta + 2/4)(\vartheta + 3/4), \quad \vartheta = z\partial_z \tag{10.9}$$

be the differential operator on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ associated to the generalized hypergeometric function ${}_3F_2(1/4, 2/4, 3/4; 1; 1; u)$ then

$$z^* {}_3\mathcal{D}_2 = \frac{1}{64}(1 - t^4).\mathcal{D}.t.$$

Proof. Direct calculation. □

Example 10.27. The function on B

$${}_3F_2(\frac{1}{4}, \frac{2}{4}, \frac{3}{4}; 1; 1; t^{-4})t$$

defined for $|t| > 1$ satisfies the Picard–Fuchs equation.

Consider the solutions to the hypergeometric differential equation ${}_3\mathcal{D}_2$

$$W_1(z) = \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4(4^4)^n} z^n = {}_3F_2(\frac{1}{4}, \frac{2}{4}, \frac{3}{4}; 1, 1; z)$$

$$W_2(z) = \ln(4^{-4}z)W_1(z) + 4\sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4(4^4)^n}[\Psi(4n + 1) - \Psi(n + 1)]z^n.$$

where Ψ denotes the digamma-function $\Psi(z) = \Gamma'(z)/\Gamma(z)$. The functions $W_i(t^{-4}), i = 1, 2$ are solutions to the pulled back equation $z^* {}_3\mathcal{D}_2$. We set

$$P(t) := \frac{1}{2\pi i} \frac{W_2(t^{-4})}{W_1(t^{-4})}.$$

These functions converge for $|t| > 1$ and hence define germs at the point $t_1 = i\sqrt{2}$. The logarithm is chosen in such a way that $Im(\ln((4t_1)^{-4})) = 0$.

Choose a path $\delta : t_0 \to t_1$, $t_0 = i/\sqrt{2}, t_1 = i\sqrt{2}$ within the contractible region $\{t \,|\, Re(t) > 0, Im(t) > 0\} \subset B$. We get an isomorphism between the fundamental groups by

$$T_\delta : \pi_1(B, t_0) \to \pi_1(B, t_1), \quad \gamma \mapsto \delta \cdot \gamma \cdot \delta^{-1}$$

The analytic continuation along $T_\delta\gamma_\infty$ can be read off the definition

$$AC_{T_\delta\gamma_\infty}W_1 = W_1, \quad AC_{T_\delta\gamma_\infty}W_2 = W_2 + 4(2\pi i)W_1.$$

Indeed, the sums define holomorphic functions and are therefore unaffected by analytic continuation. The only contribution comes from the logarithmic term. The path $T_\delta\gamma_\infty$ encircles ∞ once with positive orientation. Therefore 0 is encircled with negative orientation, so the logarithm picks up a summand $-2\pi i$.

We can apply the first part of criterion 10.25 to see

$$\mathscr{P}^c(t) = P(t) + \mu$$

as germs of functions at $\tilde{t_1} := \delta \cdot \tilde{t_0}$ for some $\mu \in \mathbb{C}$. To apply the second part of the criterion we need the following additional information.

Theorem 10.28 (Nagura, Sugiyama [NS95])**.** *An analytic continuation of the map $P := \frac{1}{2\pi i}W_2/W_1$ to a sliced neighborhood of $t = 1$ is given by*

$$\begin{aligned} P(t) &= \frac{i}{\sqrt{2}}\frac{U_1(t) + \tan(\frac{\pi}{8})^{-1}U_2(t)}{U_1(t) - \tan(\frac{\pi}{8})^{-1}U_2(t)} \\ U_1(t) &= \frac{\Gamma(\frac{1}{8})^2}{\Gamma(\frac{1}{2})}{}_2F_1(\frac{1}{8}, \frac{3}{8}; \frac{1}{2}; 1 - t^4) \\ U_2(t) &= \frac{\Gamma(\frac{5}{8})^2}{\Gamma(\frac{3}{2})}(t^4 - 1)^{1/2}{}_2F_1(\frac{5}{8}, \frac{5}{8}; \frac{3}{2}; 1 - t^4). \end{aligned}$$

Thus the monodromy around the point $t = 1$ satisfies $AC_{T_\delta\gamma_4}P = -\frac{1}{2P}$.

We find get following corollary.

Theorem 10.29. *The composition of the period map with the parametrization of the period domain (10.7)*

$$\mathscr{P}^c = \tilde{exp} \circ \mathscr{P} : \tilde{B} \longrightarrow \mathscr{D}(T_0) \longrightarrow \mathbb{C} \setminus \mathbb{R}$$

is explicitly given in a neighborhood of $\tilde{t_1}$ by

$$\mathscr{P}^c(t) = P(t) = \frac{1}{2\pi i}\frac{W_2(t^{-4})}{W_1(t^{-4})}.$$

Proof. We have to check, that the function P has the right analytic continuation along $T_\delta\gamma_1$, i.e. $AC_{T_\delta\gamma_1}P = -1/(2P)$. We know the analytic continuation of P along $T_\delta\gamma_4$ has this form.

But P only depends on $z = t^{-4}$ not on t itself. Moreover the images of the paths γ_1 and γ_4 under $t \mapsto t^{-4}$ coincide. Hence also the analytic continuations are the same. □

Proposition 10.30. *The power series expansion of $\mathscr{P}^c$ at $t = \infty$ is given by*

$$\mathscr{P}^c(w) = \frac{1}{2\pi i}(\ln(w) + 104\,w + 9780\,w^2 + 4141760/3\,w^3 + 231052570\,w^4 + \dots)$$

$$exp(2\pi i\mathscr{P}^c(w)) = w + 104\,w^2 + 15188\,w^3 + 2585184\,w^4 + 480222434\,w^5 + \dots$$

where $w = 1/(4t)^4 = 4^{-4}z$.

This is precisely the series obtained by Lian and Yau [LY96][4] using a different method (see Remark 11.2). They also prove that the expansion of $exp(2\pi i\mathscr{P}^c(w))$ has integral coefficients.

Corollary 10.31. *The period map $\mathscr{P}^c$ takes values in the upper half plane.*

Remark 10.32. We show how Theorem 1.9 stated in the introduction can be derived from 10.29.

We identify $H^2(X_{t_0}, \mathbb{Z}) \cong \Lambda$ via the isomorphism given in Theorem 10.8 and use parallel transport to extend this isomorphism to nearby fibers X_t.

The period vector Ω_t is contained in $T_0 \otimes \mathbb{C}$, where $T_0 = \langle 4 \rangle \oplus U \subset \Lambda$ is the generic transcendental lattice. By Theorem 10.29 and (10.7) we have

$$[\Omega_t] = [e\tilde{x}p(\mathscr{P}^c(t))] \in \mathcal{D}(T_0) \subset \mathbb{P}(\Lambda_\mathbb{C})$$

and hence there is a nowhere vanishing holomorphic function $f(t)$ such that

$$f(t)\Omega_t = e\tilde{x}p(\mathscr{P}^c(t)) = \mathscr{P}^c(t)h - e + 2(\mathscr{P}^c(t))^2 f. \tag{10.10}$$

As $f(t)\Omega_t$ is also a non-vanishing holomorphic two-form we can assume this equation holds true already for Ω_t. The period integrals can now be calculated as intersection products $\int_\Gamma \Omega_t = \Omega_t.\Gamma$.

The required basis Γ_i of $\Lambda = 2E_8(-1) \oplus U'' \oplus U' \oplus U$ is constructed as follows. We let $(\Gamma_1, \Gamma_2, \Gamma_3) = (h, e, f)$ be the standard basis of T_0. Recall that $h = e' + 2f'$ and hence $(\Gamma_1, \Gamma_4) = (h, f')$ is a basis of U'. The remaining basis vectors can be chosen to be any basis of the orthogonal complement $2E_8(-1) \oplus U''$ of $(\Gamma_1, \ldots, \Gamma_4)$. Using (10.10) it is now straightforward to calculate the entries of the period vector.

10.8 The period map as Schwarz triangle function

In this chapter we will relate the period map to a Schwarz triangle function. We begin by recalling some basic facts about these functions from [Beu07].

Definition 10.33. The *hypergeometric differential equation* with parameters $a, b, c \in \mathbb{C}$ is

$$\vartheta(\vartheta + c - 1)f - z(\vartheta + a)(\vartheta + b)f = 0, \quad \vartheta = z\partial_z,\ f \in \mathcal{O}_\mathbb{C} \tag{10.11}$$

which is satisfied by the hypergeometric function $f = {}_2F_1(a, b; c; z)$.

Let f, g be two independent solutions to this differential equation at a point $z_0 \in \mathbb{H}$. The function $D(z) = f/g$ considered as map $\mathbb{H} \to \mathbb{C}$ is called *Schwarz triangle function.*

These functions have very remarkable properties and were studied extensively in the 19th century (see Klein's lectures [Kle33]).

Definition 10.34. A *curvilinear triangle* is an open subset of $\mathbb{P}^1$ whose boundary is the union of three open segments of circles or lines and three points. The segments are called *edges* and the points *vertices* of the triangle.

[4]Equation 5.18 contains an expansion of the inverse series to ours.

Proposition 10.35. *For any three distinct points $A, B, C \in \mathbb{P}^1$ and positive, real numbers λ, μ, ν with $\lambda + \mu + \nu < 1$ there is a unique curvilinear triangle with vertices (A, B, C) and interior angles $(\lambda\pi, \mu\pi, \nu\pi)$ in that order.*

Theorem 10.36 (Schwarz, [Beu07] 3.20)**.** *A Schwarz triangle function maps the closed upper half plane $\mathbb{H} \cup \mathbb{R}$ isomorphically to a curvilinear triangle.*

The vertices are the points $(D(0), D(1), D(\infty))$ and the corresponding angles $(\lambda\pi, \mu\pi, \nu\pi)$ depend on the parameters of the hypergeometric differential equation via $\lambda = |1-c|, \mu = |c-a-b|, \nu = |a-b|$.

Recall that the period map is a function on the universal cover of $B = \mathbb{P}^1 \setminus \Sigma$ to the upper half plane.

$$\mathscr{P}^c : \tilde{B} \longrightarrow \mathbb{H}.$$

This maps descends along $t \mapsto z(t) = t^{-4}$ to a multi-valued map on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. We explain this last sentence more formally. The map $t \mapsto z(t) = t^{-4}$ is an unramified covering $B \setminus \{0\} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Hence it induces an isomorphism between the universal covering spaces. Moreover the inclusion $B \setminus \{0\} \to B$ induces a map $(B \setminus \{0\})^{\sim} \to \tilde{B}$. We use the composition

$$(\mathbb{P}^1 \setminus \{0, 1, \infty\})^{\sim} \cong (B \setminus \{0\})^{\sim} \longrightarrow \tilde{B}$$

to view $\mathscr{P}^c : \tilde{B} \to \mathbb{H}$ as multi-valued map on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

We choose a basepoint $\tilde{z}_1$ of $(\mathbb{P}^1 \setminus \{0, 1, \infty\})^{\sim}$ mapping to $\tilde{t_1}$. Denote by ι the unique lift of the inclusion $\mathbb{H} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ to the universal cover of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ mapping z_1 to $\tilde{z}_1$ (when extended to the boundary of $\mathbb{H} \subset B$).

Theorem 10.37. *The restriction of the period map*

$$\mathscr{P}^c(z) : (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{\sim} \longrightarrow \mathbb{P}^1$$

to $\iota : \mathbb{H} \to (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{\sim}$ is a Schwarz triangle function. The upper half plane is mapped to the triangle with vertices $(\infty, \frac{i}{2}, \frac{1+i}{2})$ and angles $(0, \pi/2, \pi/4)$ as pictured in Figure 1.2 in the introduction.

Proof. The strategy is the following. We first construct the a triangle function with the expected mapping behavior. Then we write this function as a quotient of solution of the Picard–Fuchs equation. Finally we show that the assumptions of Proposition 10.25 are satisfied by this function. It follows that it has to be the period map.

Step 1. Let f, g be two independent solutions to ${}_2\mathcal{D}_1$ at t_1. By Schwarz' theorem $D(z) = f/g$ is a triangle function. Using a Möbius transformation, we can change the vertices of the triangle to be $(0, 1, \infty)$. As the composition is again of the form f'/g' for independent solutions f', g' of ${}_2\mathcal{D}_1$ we can assume $D(z)$ maps $(0, 1, \infty)$ to $(\infty, \frac{i}{\sqrt{2}}, \frac{1+i}{2})$.

The triangle pictured in green color in Figure 1.2 is the unique curvilinear triangle with vertices $(\infty, \frac{i}{\sqrt{2}}, \frac{1+i}{2})$ and interior angles $(0, \pi/2, \pi/4)$. Hence it is the image of $\mathbb{H}$ under $D(z)$.

The analytic continuation of $D(z)$ can be obtained by reflecting the triangle at its edges. This technique is called *Schwarz reflection principle* (see [Beu07] for details).

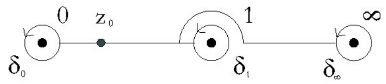


Figure 10.2: The paths δ_i in $\mathbb{P}^1 \setminus \{0,1,\infty\}$ based at $z_1 = 1/4$.

Let $\delta_0, \delta_1 \in \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}, z_1)$ be the paths pictured in Figure 10.8 encircling $0,1$ once with positive orientation respectively. Reflecting the triangles according to the crossings of the paths with the components of $\mathbb{R} \setminus \{0,1\}$ we find

$$AC_{\delta_0} D(z) = D(z) + 1, \quad AC_{\delta_1} D(z) = \frac{-1}{2D(z)}.$$

This means that $AC_{\delta_0}(f/g) = (f+g)/g$ and since f, g are independent we can conclude that there is a $\lambda \in \mathbb{C}^*$ such that

$$AC_{\delta_0}(f,g) = (f,g).\begin{pmatrix} \lambda & 0 \\ \lambda & \lambda \end{pmatrix}. \tag{10.12}$$

The hypergeometric function ${}_2F_1(\frac{1}{8}, \frac{3}{8}; 1; z)$ is a linear combination of the basis solutions (f,g). Since it is holomorphic at 0, the matrix (10.12) has to have the eigenvalue 1 which is only the case if $\lambda = 1$.

Step 2. The ${}_3F_2$-hypergeometric function $W_1(z)$ occuring in the expansion of the period map is related to a ${}_2F_1$-hypergeometric function by the Clausen identity ([Bai35], p.86)

$${}_3F_2(\frac{1}{4}, \frac{2}{4}, \frac{3}{4}; 1; 1; z) = {}_2F_1(\frac{1}{8}, \frac{3}{8}; z)^2.$$

The corresponding statement in terms of differential equations reads as follows.

Proposition 10.38. *The differential equation*

$${}_2\mathcal{D}_1 = \vartheta^2 - z(\vartheta + 1/8)(\vartheta + 3/8), \quad \vartheta = z\frac{\partial}{\partial z} \tag{10.13}$$

associated to the hypergeometric function ${}_1F_2(\frac{1}{8}, \frac{3}{8}; 1; z)$ has the property that for all solutions f, g to ${}_2\mathcal{D}_1$ the product satisfies ${}_3\mathcal{D}_2(f.g) = 0$.

Conversely any solution to ${}_3\mathcal{D}_2$ is a sum of products of solutions to ${}_2\mathcal{D}_1$.

Proof. The proposition can be rephrased by saying ${}_3\mathcal{D}_2 = Sym^2({}_2\mathcal{D}_1)$. There is an algorithm to compute such symmetric squares of differential operators, which is implemented e.g. in Maple. We used this program to verify the equality. □

Using this proposition and Proposition 10.26 we can trivially express $D(z)$ as a quotient of solutions of the Picard–Fuchs equation (10.5), namely

$$D(t^{-4}) = \frac{f(t^{-4})}{g(t^{-4})} = \frac{f(t^{-4})g(t^{-4})\,t}{g(t^{-4})^2\,t}.$$

Step 3. We claim that the tuple $(a,b) = (f(t^{-4})g(t^{-4})\,t, g(t^{-4})^2\,t)$ of solutions of the Picard–Fuchs equation satisfies the assumptions of the criterion 10.25.

The paths $T_\delta\gamma_\infty, T_\delta\gamma_1 \in \pi_1(B, t_1)$ in B map to $\delta_0^4, \delta_1 \in \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}, z_1)$ under $t \mapsto z(t) = t^{-4}$. Hence we can calculate the monodromy transformations as

$$AC_{T_\delta\gamma_\infty}(f,g) = (f,g).\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^4 = (f,g).\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$$

and consequently also

$$AC_{T_\delta\gamma_\infty}(a,b) = (a,b).\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$$

moreover

$$AC_{T_\delta\gamma_1} D(t^{-4}) = \frac{-1}{2D(t^{-4})}$$

as required. This concludes the proof of the theorem. $\square$

Chapter 11

Mirror symmetries and mirror maps

It remains to translate the above computations in the framework developed in chapter 8.

Let $X \to B$ be the Dwork Pencil and

$$\mathscr{P}_B : \tilde{B} \longrightarrow \mathscr{D}(T_0) \subset \mathscr{D}(\Lambda) \subset \mathscr{D}(\tilde{\Lambda})$$

the (B-model) period map associated to the marking, constructed in Theorem 10.8. Here $T_0 \cong \langle h \rangle \oplus U$ is the transcendental lattice of the general member of X/B.

Let $\mathcal{Y} \to \mathbb{H}$ be the family of generalized K3 structures on a quartic $Y \subset \mathbb{P}^3$ as constructed in chapter 9 and

$$\mathscr{P}_A : \mathbb{H} \longrightarrow \mathscr{D}(\langle H \rangle \oplus U) \subset \mathscr{D}(\tilde{\Lambda})$$

the A-model period map as in Proposition 9.2. Here $\langle H \rangle \oplus U$ is the lattice spanned by the class of a hyperplane H and $U \cong H^0 \oplus H^4 \subset \tilde{H}(Y, \mathbb{Z})$.

Theorem 11.1. *Mirror symmetry as described in chapter 8.10 between the symplectic quartic in $\mathbb{P}^3$ and the Dwork family is determined by the diagram*

$$\begin{tikzcd}
\tilde{B} \arrow[r, "\mathscr{P}_B"] \arrow[d, "\psi"'] & \mathscr{D}(T_0) \arrow[r] \arrow[d, "g_0"] & \mathscr{D}(\tilde{\Lambda}) \arrow[d, "g"] \\
\mathbb{H} \arrow[r, "\mathscr{P}_A"] & \mathscr{D}(\langle H \rangle \oplus U) \arrow[r] & \mathscr{D}(\tilde{\Lambda})
\end{tikzcd}$$

where $g \in \mathcal{O}(\tilde{\Lambda})$ is a isometry interchanging $H^0 \oplus H^4$ with $U \subset T_0$ and $\psi = \mathscr{P}^c$ is the period map of Theorem 10.29.

Proof. Recall from 9.2 that $\mathscr{P}_A(z) = [1e + zH - 2z^2 f]$. On the other hand $\mathscr{P}^c$ was defined using the parametrization $\tilde{exp}(z) = [-1e + zh + 2z^2 f]$. So in order for the diagram to commute we should use the isometry

$$g_0 : T_0 = (\langle h \rangle \oplus U) \longrightarrow (\langle H \rangle \oplus U),\ h \mapsto H, e \mapsto -e, f \mapsto -f$$

to relate the period domains $\mathscr{D}(T_0)$ and $\mathscr{D}((\langle H \rangle \oplus U))$. This isomorphism is easily seen to extend to an isometry g of $\tilde{\Lambda}$ using Nikulin's theorem 10.12. □

Remark 11.2. A mirror map in the sense of Morrison [Mor92] is a quotient $\psi = a/b$ of two solutions to the Picard–Fuchs equation a, b satisfying the property

$$AC_{\gamma_\infty}\psi = \psi + 1$$

for analytic continuation around the point of maximal unipotent monodromy. As in Proposition 10.25 one finds that ψ is uniquely determined up to addition of a constant. One chooses this constant in such a way that the Fourier expansion at ∞ has integral coefficients.

Such a function can be constructed directly from the differential equation by using a *Frobenius basis* for the solutions at the singular point. Using this method, Lian and Yau [LY96] arrive at precisely the same formula 10.30.

There are several differences to our definition. First note, that our mirror maps are symmetries of the period domain of (generalized) K3 surfaces which become functions only after composition with the corresponding period maps.

Secondly and more importantly, we do require the solutions a, b to be of the form $\int_\Gamma \omega_t$, for some *integral* cycle $\Gamma \in H^2(X_{t_0}, \mathbb{Z})$. It is not clear (and in general not true) that the Frobenius basis has this property. This was the main difficulty we faced above. Our solution relied heavily on the work of Narumiyah and Shiga [NS01].

There is also a conceptual explanation that Morrison's mirror map coincides with ours. Conjecturally (see [KKP08], [Iri09]) the Frobenius solutions differ form the integral periods by multiplication with the $\hat{\Gamma}$-class

$$\hat{\Gamma}(X) = \prod_{i=1}^{n} \Gamma(1+\delta_i) = exp(-\gamma c_1(X) + \sum_{k\geq 2}(-1)^k(k-1)!\zeta(k)ch_k(TX))$$

where δ_i are the Chern roots of TX, γ is Euler's constant and $\zeta(s)$ is the Riemann zeta function. The Calabi–Yau condition $c_1(X) = 0$ translates into the statement, that the first two entries of the Frobenius basis give indeed integral periods. In our case, this information suffices to fix the Hodge structure completely.

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