KLEINE AG: HONDA-TATE THEORY

Organisation:
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The theorem of Honda and Tate establishes a complete classification of isogeny classes of abelian varieties over finite fields. More precisely, one has the following

Theorem 0.1 (Main Theorem)**.** *Let $k = \mathbb{F}_q$ be a finite field with q elements and ℓ prime distinct from* $\operatorname{char} k$*. One has a bijection*

$$\mathscr{M}(k) \xrightarrow{\sim} W(q), \tag{0.1}$$

where $\mathscr{M}(k)$ denotes the set of simple abelian varieties modulo k-isogeny and $W(q)$ the set of conjugacy classes of Weil q-numbers.

Here, a Weil q-number means an algebraic integer of absolute value $q^{1/2}$ with respect to any embedding of $\bar{\mathbb{Q}}$ into $\mathbb{C}$. The map in the theorem is well-defined by the Riemann hypothesis for abelian varieties over finite fields, which was formulated and proven by Weil. The injectivity of this map follows from the general result of Tate in [Ta]:

$$\operatorname{Hom}_k(A, B) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \xrightarrow{\sim} \operatorname{Hom}_{G_k}(T_\ell(A), T_\ell(B)), \tag{0.2}$$

where G_k is the absolute Galois group of k and $\ell \neq \operatorname{char}(k)$ a prime number. Here, the injectivity holds in general. This isomorphism (0.2) has also another important applications. In particular, it provides a proof of the Tate conjecture, which states that Galois invariant étale cohomology clases come from algebraic cycles, in the special case of the first étale cohomology of abelian varieties. The surjectivity of the map in the main theorem was proven by Honda in [Ho].

Our aim is to provide a full proof of theorem 0.1. In the first two talks we prove the Tate part of the theory, i. e. the isomorphism (0.2), where the surjectivity is the far more complicated part. In the third talk we discuss some applications of the Tate isomorphism. In the talks four and five we want to give a proof of the surjectivity of the map in theorem (0.1).

First talk. Set up and injectivity of (0.2)**, first part of surjectivity.** The main reference for this talk is [Ta]. Here is a possible structure of the talk:

- Recall briefly the ℓ-adic Tate module of an abelian variety as a Galois module. Establish the map (0.2).
- Proof the injectivity of the map (0.2) (follow [Mi]).
- From now on, we assume k finite. Following [Ta], begin with the proof of surjectivity of (0.2). More precisely:
- Reduce to the rational Tate-module and its endomorphism algebra (Lemmata 1 and 3)
- Show that the injectivity is essentially independent of ℓ (Lemma 2)
- Reduce to showing the semisimplicity of the Frobenius and a commutant relation (Lemma 4)
- Introduce the crucial hypothesis $\operatorname{Hyp}(k, A, d, \ell)$ on finiteness, which is more or less trivial over finite fields and recall briefly the Weil pairing of polarized abelian varieties (beginning of §2).

Other references include [Mu], [Mi] and [CS].

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Second talk. Finish of the proof of surjectivity. The reference for this talk is [Ta], pp. 136–139.

- Assuming Hyp(k, A, d, ℓ) and the existence of a maximal isotropic Galois invariant subspace $W \subseteq V_\ell(A)$, prove the existence of a projector

$$u\colon V_\ell(A) \twoheadrightarrow W,$$

  coming from an endomorphism of A (see [Ta], Proposition 1).
- Assuming Hyp(k, A, d, ℓ) and the semi-simplicity of Frobenius over $\bar{\mathbb{Q}}_\ell$, prove the bijectivity of (0.2) (see [Ta], Proposition 2).
- Finish the proof, by showing the semi-simplicity of Frobenius and the independence of ℓ (see [Ta], end of §2).

Third talk. Applications. Here we see some applications of the Tate-isomorphism established in the first two talks. The main reference is again [Ta].

- Define Weil numbers and write down the isomorphism (0.1). (see [Oo])
- State and prove Theorem 1 of [Ta], which gives equivalent conditions for abelian varieties to be isogeneous. In particular, this gives the injectivity of the (0.1) (see also [Ei], §6, p. 9).
- State Theorem 2 of [Ta], which establish some properties of the endomorphism algebra $\mathrm{End}_k(A) \otimes \mathbb{Q}$. Prove as much as possible of it.

Fourth talk. Essential surjectivity I. The goal of this talk is begin with the proof of surjectivity of the map (0.1). The reference for this talk is [Ei].

- Give some examples of Weil numbers (§5, p. 7–8).
- A Weil number π is called *effective*, if there is a simple abelian variety $A/\mathbb{F}_q$ such that its Frobenius has π as an eigenvalue under some embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ (§7).
- Let $N \geq 1$. If π^N is effective, then so is π (Lemma 7.1).
- Define the central division algebra $E = E(\pi)$ over $F = \mathbb{Q}(\pi)$. Prove the existence of a CM-field containing F and splitting E (Lemma 7.2)
- Essentially, an abelian variety is of type (L), if it has complex multiplication by L (Definition 7.3).
- State Lemma 7.4 and conclude the proof of the Main Theorem using all these lemmata.

Fifth talk. Essential surjectivity II. The references for this talk are [Ei], [ST1] and [ST2].

In this (probably most challenging) talk we finish the proof of the main theorem by sketching the proof of Lemma 7.4 in [Ei]. The crucial construction (see [Ei], Lemma 7.6) in the proof is the existence of a certain abelian scheme over the ring of integers of some number field, with complex multiplication of type (L, Φ) (Definition 7.5). We omit this construction, which is given in [ST1] Theorem 7 and [ST2] 6.2 and 12.4. But if time permits, sketch the main idea.

The second black box is (see [Ei] Lemma 7.7) that we can lift the Frobenius from the special fibre of the constructed variety A to an endomorphism of the whole scheme coming from L, see [ST1] §7 and [Ho] p. 89.

References

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