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Submit the solutions in groups of two at the lecture on Thursday, 2018-05-03

Exercise 1. Let f be a tempered distribution. Define $\operatorname{supp} f$ to be the minimal closed set $K \subset \mathbb{R}^d$ such that $f(\varphi) = 0$ for all Schwartz functions supported in K^c.

(a) Let ψ be a Schwartz function. Prove that $(\psi f)(\varphi) = f(\psi\varphi)$ defines a tempered distribution.

(b) Find example of a tempered distribution f and a Schwartz function ψ such that $\psi(x) = 0$ for all $x \in \operatorname{supp} f$ but $\psi f \neq 0$. (Hint: consider derivatives)

(c) Let $f_n(x) = 1_{[0,1]}(x)\cos(2\pi n x)$ for $x \in \mathbb{R}$. Define $f_n(\varphi) = \int_{\mathbb{R}} f_n \varphi$ for Schwartz functions φ. Prove that f_n is a tempered distribution. Show that $\lim_{n\to\infty} f_n = 0$ as distribution but not in $L^p(\mathbb{R})$ for any $p \geq 1$.

Exercise 2. (a) Suppose that $1 + 1/r = 1/p + 1/q$, $r, p, q \geq 1$, and $f \in L^p(\mathbb{R}^d), g \in L^q(\mathbb{R}^d)$. Prove *Young's convolution inequality*

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

(b) Prove that there exists a Schwartz function φ such that $1_{B(0,1)} \leq \widehat{\varphi}(\xi) \leq 1_{B(0,2)}$.

(c) Let $f \in \mathcal{S}$ be such that $\operatorname{supp} \widehat{f} \subset B(\xi_0, R)$ for some $\xi_0 \in \mathbb{R}^d$ and $R > 0$. Prove *Bernstein's inequality*: for all $q \geq p \geq 1$

$$\|f\|_q \leq C R^{d\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_p.$$

Hint: Use the previous items and what you know about Fourier transform of translations and dilations.

Remark. If f is supported in $B(x_0, R^{-1})$, then it follows from Hölder's inequality that

$$R^{d\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_p \leq C\|f\|_q$$

for all $q \geq p \geq 1$. According to Bernstein's inequality, this behaviour is reversed in the case of compact Fourier support.